Necessary and Sufficient Conditions for Attitude Estimation in Fractionated Spacecraft Systems

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This paper addresses the problem of attitude estimation in fractionated spacecraft clusters. Each module in the cluster may have either a startracker, a relative attitude sensor, or both. Using results in nonlinear observability theory, we provide graph-theoretic sufficient conditions for the attitude of every module to be observable. In particular we show that the attitude of every module in the cluster can be observed if every module has either a star tracker with non-collinear stars, or there is a path through the sensing network from a module with a star tracker to the module without a star tracker, and each of the relative measurements along the path has either multiple non-collinear beacons or a single beacon that is not parallel to the rotation vector of the target module.

I. Introduction

During the lifecycle of a space mission, a number of forms of uncertainty can affect the design and operation of a spacecraft. Ref. 1 identifies six different types of uncertainty, which in some cases can comprise both risks and opportunities. Technical uncertainty consists of events such as an in-flight component failure or a software bug. Environmental uncertainty consists of impact with space objects, or radiation levels that are beyond their expected values. Launch uncertainty causes risk due to the possibility of launch failure, but presents opportunities in the form of the introduction of new launch vehicles. Demand uncertainty occurs because the need for a particular spacecraft's services may change dramatically after launch, for example because of competing providers. Requirements uncertainty can cause requirements to change during a design cycle. Finally, the funding stream for a given project is uncertain, since competing budget priorities can cause the available funding level to fluctuate during its lifetime. In order to provide the best possible return on investment, space missions should be responsive to uncertainty. A *responsive* space system can, for example, continue to operate after component failures, respond quickly to unforeseen threats, and scale according to the available funding.[1]

The vast majority of previous and current space missions were designed to achieve their goals using a single, monolithic spacecraft. Ref. 1 argues that such spacecraft are highly unresponsive to uncertainty. For example, conventional spacecraft design uses a high degree of redundancy to mitigate the effects of technical uncertainty, however this can greatly increase the cost and complexity of a monolithic spacecraft. Funding uncertainty can halt the development of the entire spacecraft before any services have been provided, while demand and requirements uncertainty can mean that, even if launched successfully, the monolithic spacecraft may not address the true needs of the customer.

Recent work proposed the concept of fractionated free-flying spacecraft.[1] Instead of a single, monolithic spacecraft, a fractionated free-flying spacecraft uses multiple spacecraft modules. These modules are connected only through wireless communication links and, potentially, wireless power links. The key advantage of this concept, as noted by Ref. 1, is the ability to respond to uncertainty. For example, if a single spacecraft module in the cluster fails, a new one can be launched at a lower cost and risk than would be incurred with on-orbit servicing or replacement of the monolithic spacecraft.

In order to design a fractionated spacecraft system, a number of key questions need to be answered, including the following:

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What are the capabilities of the fractionated spacecraft system, as a function of the capabilities of the individual modules?

Answering this question is critical for two reasons. First, it enables pre-launch system-level design of the fractionated spacecraft; if we can answer the above question, then we can perform trade studies between different modules and between different designs within each module. Second, it enables the designer to determine how the system will respond to uncertainty. For example, we can determine how significantly the capabilities of the overall system will degrade if a particular sensor fails, or if a module is replaced with one with different capabilities.

The navigation capabilities of the fractionated system are of particular importance. Ref. 1 notes that, while the modules in the cluster may be allowed to drift relative to each other, knowledge of the relative position and attitude of each spacecraft is essential to avoid collision and to ensure that communication and power links are maintained. Ref. 1 notes that it is not necessary, however, for each module to have the hardware necessary to determine its inertial position and attitude completely. Instead, the navigation functionality can be fractionated, so that only one spacecraft knows its inertial position and attitude with the others determining their position and attitude relative to the navigation module.

In this paper we focus our attention on the *attitude estimation* capabilities of the fractionated spacecraft system. In particular, we consider the problem of determining the attitude of *every* module using all of the hardware available to the fractionated system. This hardware includes star trackers, gyros, and relative attitude sensors. Using concepts of nonlinear observability from Ref. 2, we present necessary conditions for observability of the attitude of every module in the cluster. These conditions specify the *minimum* sensing capabilities required for the attitude of every module to be determined by an estimation algorithm; this will enable the derivation of practical estimators for fractionated attitude determination.

II. Related Work

The problem of control and estimation of spacecraft clusters has received a great deal of attention in recent years. Much of this research focused on the formation control problem, where spacecraft move in tightly coordinated formations, for example Refs. 3–9. See Ref. 10 for a review of control in spacecraft formations. The problem of *estimation* for clusters of spacecraft, and other vehicles, was considered by a number of authors, including Refs. 11–14. Much of this work deals with the estimation of relative position and velocity using relative position measurements. Refs. 7, 12, 15–18 present practical estimators to solve this problem, including experimental demonstrations with relative sensing carried out by differential Global Positioning System signals.

In the present paper, by contrast, we are interested in *attitude* estimation. The extension of position estimators to attitude estimators is not straightforward for two reasons. First, attitude kinematics and observation models are nonlinear, whereas translational dynamics and observation models are linear (in deep space) or can be linearized (if the relative separation is small). This means that the standard techniques for linear systems, such as Kalman Filtering and linear observability theory, are not directly applicable. Second, attitude is usually represented in spacecraft using the four-parameter quaternion, since all three-parameter representations are singular or discontinuous for certain attitudes.[19,20] Attempting to estimate the quaternion using a nonlinear Kalman Filter, however, leads to singularity in the covariance matrix; see Ref. 21 for further discussion on this topic. Therefore, while the nonlinear Kalman filters of Refs. 12, 17, 18 have been effective for position and velocity estimation, extending these approaches to tackle the problem of attitude estimation is not straightforward.

Analytic properties of vehicle formations with regard to observability and estimation were investigated by a number of authors.[11,13,14] Ref. 11 determined the closed-loop dynamics of the formation as a function of the observed states. Refs. 13 and 14 formulate the sensing structure of the formation as a *sensing graph* and provide graph-theoretic conditions for observability. These specify what properties the sensing graph must have for the state of every spacecraft to be observable. These are important results, since without observability we cannot hope to estimate the state; however they only apply to linear system dynamics and linear measurements, which cannot model the nonlinear dependence of sensor measurements on attitude.

Early work in estimation of attitude for a single spacecraft using Kalman Filters is reviewed by Ref. 21, and later developments are reviewed by Ref. 19. Attitude estimation for multiple spacecraft was considered by Refs. 22–27. Ref. 24 uses tight relative attitude knowledge requirements in spacecraft formations to motivate the derivation of an Unscented Kalman Filter for attitude estimation, but this estimator still determines only

the attitude of a single spacecraft using its own sensors. Refs. 22,23 develop a Kalman Filter that uses *relative* attitude measurements in order to estimate the relative attitude between two spacecraft. This work is based on a recently-developed optical relative navigation sensor known as VISNAV; the sensor is described in more detail in Ref. 28. Other optical sensors for relative attitude determination are described in Refs. 25–27. In the present paper we extend the work of Ref. 23 by analyzing the problem of attitude estimation in an arbitrary cluster of multiple spacecraft, with both relative sensors and star trackers, providing new, analytic results on observability of the attitude kinematics.

III. Background

A. Quaternion Kinematics

In this section we review attitude kinematics, using quaternion notation to represent attitude. This review follows the development in Ref. 29. Denote as $A(\mathbf{q})$ the attitude matrix mapping a vector ${}^0\mathbf{r}$ in a reference frame $\mathbb{F}^{(0)}$ to a vector ${}^1\mathbf{r}$ in another frame $\mathbb{F}^{(1)}$, such that:

$$ {}^1\mathbf{r} = A(\mathbf{q}){}^0\mathbf{r}. \tag{1} $$

The quaternion $\mathbf{q} \in \Re^4$ is a convenient representation for attitude, and is defined by:

$$ \mathbf{q} \triangleq \begin{bmatrix} \rho \\ q_4 \end{bmatrix}, \tag{2} $$

where $\rho \in \Re^3$ and $q_4 \in \Re$. For notational simplicity we define the following cross product matrix:

$$ [\mathbf{a}\times] \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \tag{3} $$

where $\mathbf{a}$ is an arbitrary vector in $\Re^3$ and a_i denotes the i'th element of $\mathbf{a}$. The quaternion is related to the attitude matrix by:

$$ A(\mathbf{q}) = \Xi^T(\mathbf{q})\Psi(\mathbf{q}), \tag{4} $$

where:

$$ \Xi(\mathbf{q}) \triangleq \begin{bmatrix} q_4 I_{3\times 3} + [\rho\times] \\ -\rho^T \end{bmatrix} \qquad \Psi(\mathbf{q}) \triangleq \begin{bmatrix} q_4 I_{3\times 3} - [\rho\times] \\ -\rho^T \end{bmatrix}. \tag{5} $$

Successive rotations between frames can be performed by quaternion multiplication, such that:

$$ A(\mathbf{q}^{(2)})A(\mathbf{q}^{(1)}) = A(\mathbf{q}^{(2)} \otimes \mathbf{q}^{(1)}), \tag{6} $$

where $\otimes$ is the quaternion multiplication operator. The quaternion multiplication of $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ is given by:

$$ \mathbf{q}^{(2)} \otimes \mathbf{q}^{(1)} \triangleq [\Psi(\mathbf{q}^{(2)})\ \mathbf{q}^{(2)}]\mathbf{q}^{(1)} = [\Xi(\mathbf{q}^{(1)})\ \mathbf{q}^{(1)}]\mathbf{q}^{(2)}, \tag{7} $$

while the inverse of a quaternion is given by:

$$ \mathbf{q}^{-1} \triangleq \begin{bmatrix} -\rho \\ q_4 \end{bmatrix}. \tag{8} $$

Quaternion kinematics are given by:

$$ \dot{\mathbf{q}} = \frac{1}{2}\Xi(\mathbf{q})\omega = \frac{1}{2}\Omega(\omega)\mathbf{q}, \tag{9} $$

where ω is the angular velocity of the frame $\mathbb{F}^{(1)}$ defined in frame $\mathbb{F}^{(1)}$. We define:

$$\Omega(\omega) \triangleq \begin{bmatrix} -[\omega\times] & \omega \\ -\omega^T & 0 \end{bmatrix} \qquad \Gamma(\omega) \triangleq \begin{bmatrix} [\omega\times] & \omega \\ -\omega^T & 0 \end{bmatrix}. \tag{10}$$

Define a difference quaternion $\Delta\mathbf{q}$ between two quaternions $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ as:

$$\Delta\mathbf{q} \triangleq \mathbf{q}^{(2)} \otimes \mathbf{q}^{(1)-1} = \begin{bmatrix} \Delta\rho \\ \Delta q_4 \end{bmatrix}, \tag{11}$$

where $\mathbf{q}^{(1)}$ rotates frame $\mathbb{F}^{(0)}$ to frame $\mathbb{F}^{(1)}$ and $\mathbf{q}^{(2)}$ rotates frame $\mathbb{F}^{(0)}$ to frame $\mathbb{F}^{(2)}$. Then the kinematics of the difference quaternion can be shown to be:[23]

$$\Delta\dot{\mathbf{q}} = -\begin{bmatrix} [\omega^{(1)}\times]\Delta\rho \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} (\omega^{(2)} - \omega^{(1)}) \\ 0 \end{bmatrix} \otimes \Delta\mathbf{q}, \tag{12}$$

where $\omega^{(1)}$ is the angular velocity of $\mathbb{F}^{(1)}$ expressed in frame $\mathbb{F}^{(1)}$ and $\omega^{(2)}$ is the angular velocity of $\mathbb{F}^{(2)}$ expressed in frame $\mathbb{F}^{(2)}$.

B. Definition of Fractionated Spacecraft Cluster

In this section we define a fractionated spacecraft cluster for the purposes of fractionated attitude estimation. A fractionated cluster $\mathcal{F}$ consists of N spacecraft modules denoted $\mathcal{S}^{(1)}$ through $\mathcal{S}^{(N)}$. The cluster has a reference coordinate frame, denoted $\mathbb{F}^{(0)}$, which has its origin at ${}^I\mathbf{x}_0$ in the inertial frame. We use ${}^i\mathbf{r}$ to denote an arbitrary vector $\mathbf{r}$ expressed in frame $\mathbb{F}^{(i)}$, and use ${}^I\mathbf{r}$ to denote a vector in the inertial frame. The reference coordinate frame is defined relative to the inertial frame such that for an arbitrary vector $\mathbf{r}$:

$${}^I\mathbf{r} = {}^I\mathbf{x}_0 + A(\mathbf{q}_0^{-1})^0\mathbf{r}. \tag{13}$$

Each module definition $\mathcal{S}^{(i)}$ consists of the following parameters:

- Body-fixed coordinate frame, denoted $\mathbb{F}^{(i)}$.
- Module orbit, denoted $\mathcal{O}^{(i)}$, consisting of the position ${}^0\mathbf{p}^{(i)}(t)$ and velocity ${}^0\mathbf{v}^{(i)}(t)$ of the origin of $\mathbb{F}^{(i)}$ in the frame $\mathbb{F}^{(0)}$.
- Module angular velocity, denoted ${}^i\omega^{(i)}$, expressed in the body-fixed frame $\mathbb{F}^{(i)}$.
- Star tracker set $\mathcal{T}^{(i)}$.
- Relative sensor set $\mathcal{R}^{(i)}$.

The star tracker set $\mathcal{T}^{(i)}$ consists of $n_{\mathcal{T}^{(i)}}$ members, each defining a star tracker. The relative sensor set consists of $n_{\mathcal{R}^{(i)}}$ relative sensors, each member defining the target module, and the locations of the target beacons in the body frame of the target module. The quaternion for the attitude of module i is denoted $\mathbf{q}^{(i)}$. The quaternion kinematics of module i are independent of the kinematics of the other modules, and are given by:

$$\dot{\mathbf{q}}^{(i)} = \frac{1}{2}\Omega({}^i\omega^{(i)})\mathbf{q}^{(i)}. \tag{14}$$

Given a fractioned cluster $\mathcal{F}$ we define the *sensing network* $\mathcal{N}(\mathcal{F})$ as a directed graph where node i represents module i, and an arc from node i to node j means that there is a relative sensor on i that has a beacon on module j as its target. A *startracker* node represents a module with one or more star trackers.

Definition 1. *A* path *from node i to node j exists if, starting with node i, it is possible to traverse arcs in the direction of the arc until node j is reached.*

The graph representation of a fractioned cluster is illustrated in Figure 1.

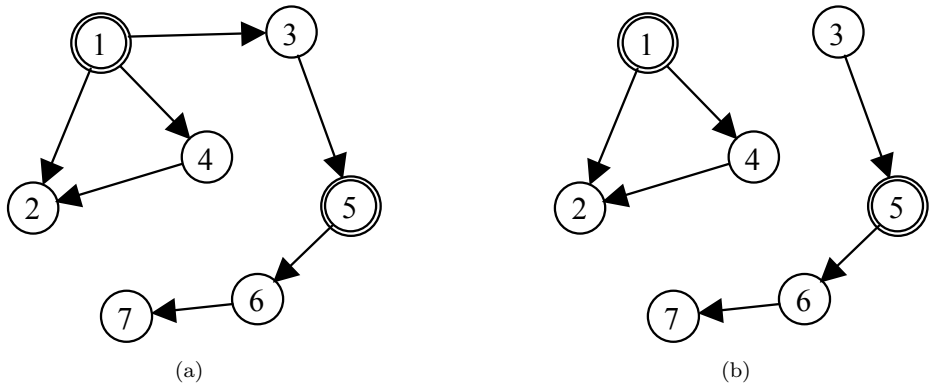

Figure 1. Directed graph representations of two different fractionated sensing networks. Arcs represent relative attitude sensing, while double circles represent modules with star trackers. In a) a path exists from node 1 to node 7, but not from node 7 to node 1. In b) no path exists from node 1 to node 7.

C. Star Tracker Sensor Model

In this paper we use the star tracker model of Ref. 29. For notational convenience we assume that each star tracker in the cluster is given a unique identifying integer from 1 to $n_{\mathcal{T}}$, where $n_{\mathcal{T}} \triangleq \sum_{i=0}^{N} n_{\mathcal{T}^{(i)}}$ is the total number of star trackers in the cluster. Star tracker i is on module l_i and makes observations of $n_{\mathbf{b},i}$ stars. The origin of star-tracker i's body-fixed coordinate frame is at ${}^{l_i}\mathbf{a}^{(i)}$ in frame $\mathbb{F}^{(l_i)}$. To simplify notation we assume that the star tracker frame is aligned with the body frame $\mathbb{F}^{(l_i)}$ of module l_i, without loss of generality. This is possible because the rotation between the body frame of a module and a star tracker on that module is fixed in the frame of the module and known. Using ${}^{0}\mathbf{s}_{i,j}$ to denote the unit vector pointing in the reference frame from the origin of star tracker frame i to star j, the star tracker model is:

$$\mathbf{b}_{i,j} = A(\mathbf{q}^{(l_i)})^{0}\mathbf{s}_{i,j}, \tag{15}$$

where $A(\mathbf{q}^{(l_i)})$ is the matrix describing the rotation from the reference frame to the body frame of module l_i. The star tracker is illustrated in Figure 2.

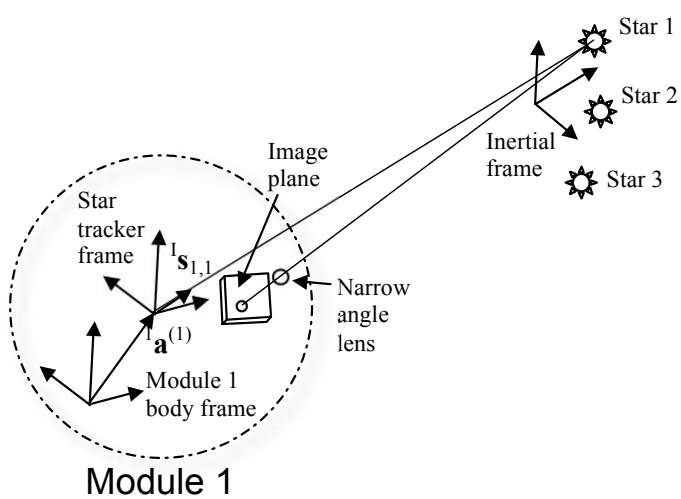

Figure 2. Star tracker model. The unit vector in the reference frame from the star tracker frame to star 1 is denoted ${}^{0}\mathbf{s}_1$, and is known from star tables. The star tracker uses a visual sensor to observe the location of the star's image on the image plane.

D. Relative Sensor Model

In this paper we consider relative sensors that detect a target point, which we refer to as a beacon, on a target module, and give either bearing, or range-and-bearing measurements. Sensors that give bearing

measurements only include the optical VISNAV sensor of Ref. 23. This sensor detects optical beacons on a target module and determines their location in the image plane. Sensors that give range-and-bearing measurements include LIDAR sensors, see for example Ref. 30 and the references therein. The beacons have known locations in the frame of the target module. Each relative sensor in the cluster is given a unique identifying integer from 1 to $n_{\mathcal{R}}$, where $n_{\mathcal{R}} \triangleq \sum_{i=0}^{N} n_{\mathcal{R}^{(i)}}$ is the total number of relative sensors in the cluster. The origin of relative sensor i's body-fixed coordinate frame is at ${}^{l_i}\mathbf{z}^{(i)}$ in frame $\mathbb{F}^{(l_i)}$. To simplify notation we assume that the relative sensor frame is aligned with the body frame $\mathbb{F}^{(l_i)}$ of module l_i, without loss of generality. This is possible because the rotation between the body frame of a module and relative sensor frame on that module is fixed in the frame of the module and known. Each relative sensor i is on module u_i and makes observations of a single beacon on module v_i. This beacon is fixed at ${}^{v_i}\mathbf{w}_i$ in the body frame $\mathbb{F}^{(v_i)}$ of the target module v_i, and we assume that $\|{}^{v_i}\mathbf{w}_i\| > 0$. We assume that each sensor has a single beacon to simplify notation, without loss of generality. We use ${}^{l_i}\mathbf{r}_i$ to denote the vector between the relative sensor and its beacon in the frame $\mathbb{F}^{(l_i)}$. The relative sensor observation for sensor i is denoted $\mathbf{c}_i$. For a range-and-bearing measurement:

$$\mathbf{c}_i = {}^{l_i}\mathbf{r}_i, \tag{16}$$

while for a bearing-only measurement:

$$\mathbf{c}_i = \frac{{}^{l_i}\mathbf{r}_i}{\|{}^{l_i}\mathbf{r}_i\|}. \tag{17}$$

The relative sensor model is illustrated in Figure 3.

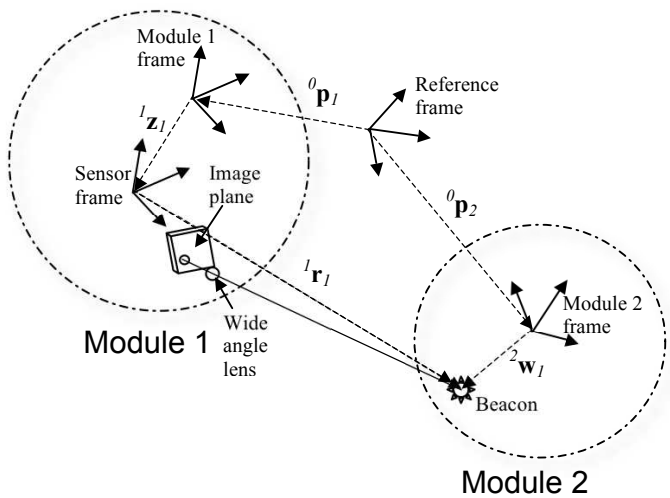

Figure 3. Relative sensor model. Shown is an optical system that gives bearing information only.

IV. Observability in Fractionated Attitude Estimation

In this section we consider the observability of the cluster attitude state. The cluster attitude state consists of the quaternion for each spacecraft, and is defined by:

$$\mathbf{x} \triangleq \begin{bmatrix} \mathbf{q}^{(1)} \\ \vdots \\ \mathbf{q}^{(N)} \end{bmatrix}. \tag{18}$$

The attitude state for the system $\Gamma(\mathcal{N})$ is in the state space $\mathcal{S}(\mathcal{N})$:

$$\mathcal{S}(\mathcal{N}) \triangleq Q^N, \tag{19}$$

where Q is the space of valid quaternions:

$$Q \triangleq \{\mathbf{x} | \mathbf{x} \in \Re^4, ||\mathbf{x}|| = 1\}. \tag{20}$$

We assume that the angular rate of each spacecraft is known and constant in the spacecraft body-fixed frame. The angular rate can be measured using gyro sensors. While such measurements are subject to gyro bias and noise, estimation of these parameters along with the spacecraft attitude is beyond the scope of this paper. We assume that all relative sensors give range-and-bearing measurements. We now define what is meant by observability in a nonlinear dynamic system.

A. Observability in General Nonlinear Systems

Consider the general nonlinear system, denoted Σ:

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}, \mathbf{u}) \\ \mathbf{y} &= g(\mathbf{x}), \end{aligned} \tag{21}$$

where $\mathbf{u} \in \Omega \subset \Re^k$, $\mathbf{x} \in M \subset \Re^n$ and $\mathbf{y} \in \Re^m$. Assume that for every bounded measurable input $\mathbf{u}(t)$ and every $\mathbf{x}_0 \in M$ there exists a unique solution to $\dot{\mathbf{x}} = f(\mathbf{x}(t), \mathbf{u}(t))$ such that $x(0) = \mathbf{x}_0$ and $\mathbf{x}(t) \in M$ for all $t \in \Re$. The following definitions are due to Ref. 2:

Definition 2. *A pair of points ξ_A and ξ_B are called **U-distinguishable** if there exists a measurable bounded input $\mathbf{u}(t)$ defined on the interval $[0, T]$ that generates solutions $\mathbf{x}_A(t)$ and $\mathbf{x}_B(t)$ of $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$ satisfying $\mathbf{x}_A(0) = \xi_A$ and $\mathbf{x}_B(0) = \xi_B$ such that $\mathbf{x}_A(t) \in U$ and $\mathbf{x}_B(t) \in U$ for all $t \in [0, T]$ and $g(\mathbf{x}_A(t)) \neq g(\mathbf{x}_B(t))$ for some $t \in [0, T]$. We use $I(\mathbf{x}_A, U)$ to denote all points $\mathbf{x}_B \in U$ that are not U-distinguishable from $\mathbf{x}_A$.*

Definition 3. *The system Σ is **observable** at $\mathbf{x} \in M$ if $I(\mathbf{x}, M) = \mathbf{x}$.*

Intuitively, this means that a system is observable at $\mathbf{x} \in M$ if for every other initial condition in M the observed time sequence is different *somewhere* on the interval $[0, T]$. Conversely, the system cannot be observable if there exists *any* other initial condition in M such that the observed time sequence is the same *everywhere* on the interval $[0, T]$. In the next sections we use this definition of observability to derive results relating to attitude estimation in fractionated spacecraft systems.

B. Sufficient Conditions for Observability with One Module

In this section, we consider the problem of attitude estimation with one module, which has a single star tracker. Then the cluster attitude state is simply:

$$\mathbf{x} \triangleq \left[\mathbf{q}^{(1)}\right]. \tag{22}$$

and the attitude kinematics are given by:

$$\dot{\mathbf{x}} = \left[\dot{\mathbf{q}}^{(1)}\right] = f(\mathbf{x}, \mathbf{u}) = f(\mathbf{x}) = \left[\tfrac{1}{2}\Omega({}^1\omega^{(1)})\mathbf{q}^{(1)}\right]. \tag{23}$$

The observations can be written in the form of (21) as follows:

$$\mathbf{y} = g(\mathbf{x}) = \begin{bmatrix} A(\mathbf{q}^{(1)}){}^0\mathbf{s}_{1,1} \\ \vdots \\ A(\mathbf{q}^{(1)}){}^0\mathbf{s}_{1,n_\mathcal{T}} \end{bmatrix}. \tag{24}$$

Using the definition of nonlinear observability given in Definition 3, we know prove the well-known result that two non-collinear ${}^0\mathbf{s}_{1,i}$ are sufficent to ensure observability of the nonlinear dynamic system described by (23) and (24).

Lemma 1. *If any pair of star vectors ${}^0\mathbf{s}_{1,i}$ and ${}^0\mathbf{s}_{1,j}$ has ${}^0\mathbf{s}_{1,i} \times {}^0\mathbf{s}_{1,j} \neq 0$, then the system described by (23) and (24) is observable at any $\mathbf{x} \in \mathcal{S}(\mathcal{N})$.*

Proof: To show observability, it suffices to show that $g(\mathbf{x}_A(0)) = g(\mathbf{x}_B(0))$ implies that $\mathbf{x}_A(0) = \mathbf{x}_B(0)$, for any $\mathbf{x}_A(0) \in \mathcal{S}(\mathcal{N})$. Let us pick i and j such that the star vectors ${}^0\mathbf{s}_{1,i}$ and ${}^0\mathbf{s}_{1,j}$ have ${}^0\mathbf{s}_{1,i} \times {}^0\mathbf{s}_{1,j} \neq 0$. The condition $g(\mathbf{x}_A(0)) = g(\mathbf{x}_B(0))$ implies that $A(\mathbf{q}_A^{(1)}(0))^0\mathbf{s}_{1,i} = A(\mathbf{q}_B^{(1)}(0))^0\mathbf{s}_{1,i}$ and $A(\mathbf{q}_A^{(1)}(0))^0\mathbf{s}_{1,j} = A(\mathbf{q}_B^{(1)}(0))^0\mathbf{s}_{1,j}$. Since $A(\mathbf{q}_B^{(1)}(0))$ has a unique inverse given by $A(\mathbf{q}_B^{(1)-1}(0))$, this implies that:

$$\begin{aligned} A\big(\mathbf{q}_B^{(1)-1}(0)\big)A\big(\mathbf{q}_A^{(1)}(0)\big)^0\mathbf{s}_{1,i} = {}^0\mathbf{s}_{1,i} \quad &\Longrightarrow \quad A\big(\Delta\mathbf{q}(0)\big)^0\mathbf{s}_{1,i} = {}^0\mathbf{s}_{1,i} \\ A\big(\mathbf{q}_B^{(1)-1}(0)\big)A\big(\mathbf{q}_A^{(1)}(0)\big)^0\mathbf{s}_{1,j} = {}^0\mathbf{s}_{1,j} \quad &\Longrightarrow \quad A\big(\Delta\mathbf{q}(0)\big)^0\mathbf{s}_{1,j} = {}^0\mathbf{s}_{1,j}, \end{aligned} \tag{25}$$

where $\Delta\mathbf{q} \triangleq \mathbf{q}_B^{(1)-1}(0) \otimes \mathbf{q}_A^{(1)}(0)$, and where we define:

$$\Delta\mathbf{q} = \begin{bmatrix} \Delta\rho \\ \Delta q_4 \end{bmatrix}. \tag{26}$$

Equation (25) implies that the star vectors ${}^0\mathbf{s}_{1,i}$ and ${}^0\mathbf{s}_{1,i}$ must each be an eigenvector of the rotation matrix $A\big(\Delta\mathbf{q}\big)$ with eigenvalue unity. This is the case if $\Delta\rho = 0$, in which case $A\big(\Delta\mathbf{q}\big)$ is the identity matrix. If $\Delta\rho \neq 0$ then $\Delta\rho$ must be parallel to both ${}^0\mathbf{s}_{1,i}$ and ${}^0\mathbf{s}_{1,j}$ since non-identity rotation matrices have exactly one eigenvector with eigenvalue unity, corresponding to the Euler axis of the rotation. However, since ${}^0\mathbf{s}_{1,i} \times {}^0\mathbf{s}_{1,j} \neq 0$ with $\|{}^0\mathbf{s}_{1,i}\| > 0$ and $\|{}^0\mathbf{s}_{1,i}\| > 0$, this is not possible. Hence $\Delta\rho = 0$, which implies that $\mathbf{x}_A(0) = \mathbf{x}_B(0)$. From Definition 2 with $U = \mathcal{S}(\mathcal{N})$, this means that $I(\mathbf{x}, \mathcal{S}(\mathcal{N})) = \mathbf{x}$. Hence from Definition 3 the system described by (23) and (24) is observable at any $\mathbf{x} \in \mathcal{S}(\mathcal{N})$. □

C. Necessary and Sufficient Conditions for Observability with Two Modules

In this section, we consider the problem of attitude estimation with two modules, i.e. $N = 2$. Then the cluster attitude state is:

$$\mathbf{x} \triangleq \begin{bmatrix} \mathbf{q}^{(1)} \\ \mathbf{q}^{(2)} \end{bmatrix}. \tag{27}$$

We can write the cluster dynamics in the form of (21) as follows:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{q}}^{(1)} \\ \dot{\mathbf{q}}^{(2)} \end{bmatrix} = f(\mathbf{x}, \mathbf{u}) = f(\mathbf{x}) = \begin{bmatrix} \frac{1}{2}\Omega({}^1\omega^{(1)})\mathbf{q}^{(1)} \\ \frac{1}{2}\Omega({}^2\omega^{(2)})\mathbf{q}^{(2)} \end{bmatrix}. \tag{28}$$

First, consider the case where there is a star tracker on module 1 and a number of relative sensors on module 1 giving range-and-bearing measurements between the sensor and beacons on module 2. Then the observations can be written in the form of (21) as follows:

$$\mathbf{y} = g(\mathbf{x}) = \begin{bmatrix} A(\mathbf{q}^{(1)})^0\mathbf{s}_{1,1} \\ \vdots \\ A(\mathbf{q}^{(1)})^0\mathbf{s}_{1,n_\mathcal{T}} \\ {}^1\mathbf{r}_1 \\ \vdots \\ {}^1\mathbf{r}_{n_\mathcal{R}} \end{bmatrix}. \tag{29}$$

We now give some analytic results relating to the observability of the system defined by (28) and (29). First, note that the observation ${}^{u_i}\mathbf{r}_i$ from the relative sensor on module u_i making measurements of a beacon on module v_i can be rewritten as:

$$\begin{aligned} {}^{u_i}\mathbf{r}_i =& A(\mathbf{q}^{(u_i)})^0\mathbf{r}_i & (30) \\ =& A(\mathbf{q}^{(u_i)})\Big({}^0\mathbf{p}_{v_i} + A(\mathbf{q}^{(v_i)-1})^{v_i}\mathbf{w}_i - {}^0\mathbf{p}_{u_i} - A(\mathbf{q}^{(u_i)-1})^{u_i}\mathbf{z}_i\Big) & (31) \\ =& A(\mathbf{q}^{(u_i)})({}^0\mathbf{p}_{v_i} - {}^0\mathbf{p}_{u_i}) + A(\mathbf{q}^{(u_i)})A(\mathbf{q}^{(v_i)-1})^2\mathbf{w}_i - {}^{u_i}\mathbf{z}_i & (32) \end{aligned}$$

Lemma 2. *If any pair of star vectors (i,j) has ${}^0\mathbf{s}_{1,i} \times {}^0\mathbf{s}_{1,j} \neq 0$ and any pair of relative sensors (k,l) has ${}^2\mathbf{w}_k \times {}^2\mathbf{w}_l \neq 0$ then the system defined by (28) and (29) is observable at any $\mathbf{x} \in \mathcal{S}(\mathcal{N})$.*

Proof: Again, to show observability, it suffices to show that $g(\mathbf{x}_A(0)) = g(\mathbf{x}_B(0))$ implies that $\mathbf{x}_A(0) = \mathbf{x}_B(0)$, for any $\mathbf{x}_A(0) \in \mathcal{S}(\mathcal{N})$. Define:

$$\mathbf{x}_A(t) = \begin{bmatrix} \mathbf{q}_A^{(1)}(t) \\ \mathbf{q}_A^{(2)}(t) \end{bmatrix} \quad \mathbf{x}_B(t) = \begin{bmatrix} \mathbf{q}_B^{(1)}(t) \\ \mathbf{q}_B^{(2)}(t) \end{bmatrix}. \tag{33}$$

The equality $g(\mathbf{x}_A(0)) = g(\mathbf{x}_B(0))$ implies that $A(\mathbf{q}_A^{(1)}(0)){}^0\mathbf{s}_{1,i} = A(\mathbf{q}_B^{(1)}(0)){}^0\mathbf{s}_{1,i}$ for all i, and following the proof of Lemma 1, since we have two star vectors with ${}^0\mathbf{s}_{1,i} \times {}^0\mathbf{s}_{1,j} \neq 0$ this implies that $\mathbf{q}_A^{(1)}(0) = \mathbf{q}_B^{(1)}(0)$. Looking now at the relative sensor measurements, $g(\mathbf{x}_A(0)) = g(\mathbf{x}_B(0))$ implies that:

$$\begin{aligned} &A(\mathbf{q}_A^{(1)}(0))({}^0\mathbf{p}_2 - {}^0\mathbf{p}_1) + A(\mathbf{q}_A^{(1)}(0))A(\mathbf{q}_A^{(2)-1}(0)){}^2\mathbf{w}_i \\ &= A(\mathbf{q}_B^{(1)}(0))({}^0\mathbf{p}_2 - {}^0\mathbf{p}_1) + A(\mathbf{q}_B^{(1)}(0))A(\mathbf{q}_B^{(2)-1}(0)){}^2\mathbf{w}_i \quad \forall i, \end{aligned} \tag{34}$$

and since $\mathbf{q}_A^{(1)}(0) = \mathbf{q}_B^{(1)}(0)$, this implies that:

$$A(\mathbf{q}_A^{(2)-1}(0)){}^2\mathbf{w}_i = A(\mathbf{q}_B^{(2)-1}(0)){}^2\mathbf{w}_i \iff A(\Delta\mathbf{q}^{(2)}){}^2\mathbf{w}_i = {}^2\mathbf{w}_i \quad \forall i, \tag{35}$$

where $\Delta\mathbf{q}^{(2)} \triangleq \mathbf{q}_B^{(2)}(0) \otimes \mathbf{q}_A^{(2)-1}(0)$ and where we have used the fact that attitude matrices have unique inverses. We define:

$$\Delta\mathbf{q}^{(2)} = \begin{bmatrix} \Delta\rho^{(2)} \\ \Delta q_4^{(2)} \end{bmatrix}. \tag{36}$$

As with the proof of Lemma 1, this implies that either $\Delta\rho^{(2)} = 0$ or $\Delta\rho^{(2)}$ is parallel to $\mathbf{w}_i$ for all i. However since there exists a pair of relative sensors (k,l) such that ${}^2\mathbf{w}_k \times {}^2\mathbf{w}_l \neq 0$, and since both $\|{}^2\mathbf{w}_k\| > 0$ and $\|{}^2\mathbf{w}_l\| > 0$, we know that $\Delta\rho^{(2)}$ is not parallel to $\mathbf{w}_i$ for all i. Hence $\Delta\rho^{(2)} = 0$, and so $\mathbf{q}_A^{(2)}(0) = \mathbf{q}_B^{(2)}(0)$ and $\mathbf{q}_A(0) = \mathbf{q}_B(0)$. From Definition 2 with $U = \mathcal{S}(\mathcal{N})$, this means that $I(\mathbf{x}, \mathcal{S}(\mathcal{N})) = \mathbf{x}$. Hence from Definition 3 the system described by (28) and (29) is observable at any $\mathbf{x} \in \mathcal{S}(\mathcal{N})$. □

Lemma 2 provides a theoretical proof of the intuitive result, that if one module has a star tracker with non-collinear stars and there is a relative sensor between the modules with non-collinear beacons, then the attitude of both modules can be determined. Now consider the case where there is a star tracker on module 1, but that there is a *single* relative sensor give range and bearing measurements to a *single* beacon on module 2. We will show that even in this case the attitude of both modules can be determined. The observations are:

$$\mathbf{y} = g(\mathbf{x}) = \begin{bmatrix} A(\mathbf{q}^{(1)}){}^0\mathbf{s}_{1,1} \\ \vdots \\ A(\mathbf{q}^{(1)}){}^0\mathbf{s}_{1,n_\mathcal{T}} \\ {}^1\mathbf{r}_1 \end{bmatrix}. \tag{37}$$

Lemma 3 gives necessary and sufficient conditions for the system described by (28) and (37) to be observable.

Lemma 3. *Assume that there exists a pair of star vectors (i,j) such that ${}^0\mathbf{s}_{1,i} \times {}^0\mathbf{s}_{1,j} \neq 0$. Then at any $\mathbf{x} \in \mathcal{S}(\mathcal{N})$ the system described by (28) and (37) is observable if and only if ${}^2\omega^{(2)} \times {}^2\mathbf{w}_1 \neq 0$.*

Proof: We first prove the 'only if' part of the lemma, that is, we show that the system is not observable if ${}^2\omega^{(2)} \times {}^2\mathbf{w}_1 = 0$. To do so it suffices to show that for any initial state $\mathbf{x}_A(0) \in \mathcal{S}(\mathcal{N})$ there exists another initial state $\mathbf{x}_A(0) \neq \mathbf{x}_B(0)$ such that $g(\mathbf{x}_A(t)) = g(\mathbf{x}_B(t))$ $\forall t \geq 0$. Define:

$$\mathbf{x}_A(t) = \begin{bmatrix} \mathbf{q}_A^{(1)}(t) \\ \mathbf{q}_A^{(2)}(t) \end{bmatrix} \quad \mathbf{x}_B(t) = \begin{bmatrix} \mathbf{q}_B^{(1)}(t) \\ \mathbf{q}_B^{(2)}(t) \end{bmatrix}. \tag{38}$$

and let $\mathbf{q}_B^{(1)}(0) = \mathbf{q}_A^{(1)}(0)$ and $\mathbf{q}_B^{(2)}(0) = \Delta\mathbf{q}^{(2)} \otimes \mathbf{q}_A^{(2)}(0)$, where $\Delta\mathbf{q}^{(2)}$ is as defined in (36). If we now choose $\Delta\rho^{(2)}(0)$ to be nonzero with $\Delta\rho^{(2)}(0) \times {}^2\mathbf{w}_1 = 0$, using the difference kinematics (12) we have:

$$\begin{aligned}
\Delta\dot{\mathbf{q}}^{(2)}(0) &= -\begin{bmatrix} [{}^2\omega^{(2)}\times]\Delta\rho^{(2)}(0) \\ 0 \end{bmatrix} = 0 \\
\Longrightarrow &\Delta\dot{\mathbf{q}}^{(2)}(t) = 0 \quad \forall t \\
\Longrightarrow &\Delta\rho^{(2)}(t) \times {}^2\mathbf{w}_1 = 0 \quad \forall t
\end{aligned} \tag{39}$$

where we have used the fact that ${}^2\omega^{(2)} \times {}^2\mathbf{w}_1 = 0$. The relative sensor measurement with initial state $\mathbf{x}_A(0)$ is given by:

$${}^1\mathbf{r}_1(\mathbf{x}_A(t)) = A(\mathbf{q}_A^{(1)}(t))({}^0\mathbf{p}_2 - {}^0\mathbf{p}_1) + A(\mathbf{q}_A^{(1)}(t))A(\mathbf{q}_A^{(2)-1}(t))^2\mathbf{w}_1 - {}^1\mathbf{z}_1, \tag{40}$$

while the relative sensor measurement with initial state $\mathbf{x}_B(0)$ is given by:

$$\begin{aligned}
{}^1\mathbf{r}_1(\mathbf{x}_B(t)) &= A(\mathbf{q}_B^{(1)}(t))({}^0\mathbf{p}_2 - {}^0\mathbf{p}_1) + A(\mathbf{q}_B^{(1)}(t))A(\mathbf{q}_B^{(2)-1}(t))^2\mathbf{w}_1 - {}^1\mathbf{z}_1 \\
&= A(\mathbf{q}_A^{(1)}(t))({}^0\mathbf{p}_2 - {}^0\mathbf{p}_1) + A(\mathbf{q}_A^{(1)}(t))A(\mathbf{q}_B^{(2)-1}(t))^2\mathbf{w}_1 - {}^1\mathbf{z}_1 \\
&= A(\mathbf{q}_A^{(1)}(t))({}^0\mathbf{p}_2 - {}^0\mathbf{p}_1) + A(\mathbf{q}_A^{(1)}(t))A(\mathbf{q}_A^{(2)-1}(t))A(\Delta\mathbf{q}^{(2)-1}(t))^2\mathbf{w}_1 - {}^1\mathbf{z}_1.
\end{aligned} \tag{41}$$

Since $\Delta\rho^{(2)}(t) \times {}^2\mathbf{w}_1 = 0$ with $\|\Delta\rho^{(2)}(t)\| > 0$ and $\|{}^2\mathbf{w}_1\| > 0$ we know that ${}^2\mathbf{w}_1$ is the Euler axis of the rotation described by $A(\Delta\mathbf{q}^{(2)-1})$, and hence $A(\Delta\mathbf{q}^{(2)-1}(t))^2\mathbf{w}_1 = {}^2\mathbf{w}_1$. Hence:

$${}^1\mathbf{r}_1(\mathbf{x}_B(t)) = A(\mathbf{q}_A^{(1)}(t))({}^0\mathbf{p}_2 - {}^0\mathbf{p}_1) + A(\mathbf{q}_A^{(1)}(t))A(\mathbf{q}_A^{(2)-1}(t))^2\mathbf{w}_1 - {}^1\mathbf{z}_1 = {}^1\mathbf{r}_1(\mathbf{x}_A(t)) \quad \forall t. \tag{42}$$

For all time, therefore, the relative measurements are the same for $\mathbf{x}_B(t)$ as for $\mathbf{x}_A(t)$. The star tracker measurements are the same since they depend only on $\mathbf{q}^{(1)}$, which is identical in $\mathbf{x}_B(t)$ and $\mathbf{x}_A(t)$. Hence $g(\mathbf{x}_A(t)) = g(\mathbf{x}_B(t))\ \forall t \geq 0$. Referring to Definition 2 and setting $M = U = \mathcal{S}(\mathcal{N})$, we see that this means that for any $\mathbf{x}_A(0)$ there exists an $\mathbf{x}_B(0) \neq \mathbf{x}_A(0)$ and $\mathbf{x}_B(0) \in I(\mathbf{x}_A(0), U)$. Hence, from Definition 3, the system defined by (28) and (37) is not observable at any $\mathbf{x} \in \mathcal{S}(\mathcal{N})$

We now prove the 'if' part of the lemma, that is, we show that the system is observable if ${}^2\omega^{(2)} \times {}^2\mathbf{w}_1 \neq 0$. It suffices to show that $g(\mathbf{x}_A(t)) = g(\mathbf{x}_B(t)) \quad \forall t \geq 0$ implies that $\mathbf{x}_A(0) = \mathbf{x}_B(0)$, for any $\mathbf{x}_A(0) \in \mathcal{S}(\mathcal{N})$. The equality $g(\mathbf{x}_A(0)) = g(\mathbf{x}_B(0))$ implies that $A(\mathbf{q}_A^{(1)}(0))^0\mathbf{s}_{1,i} = A(\mathbf{q}_B^{(1)}(0))^0\mathbf{s}_{1,i}$ for all i, and following the proof of Lemma 1, since we have two star vectors with ${}^0\mathbf{s}_{1,i} \times {}^0\mathbf{s}_{1,j} \neq 0$ this implies that $\mathbf{q}_A^{(1)}(0) = \mathbf{q}_B^{(1)}(0)$. Looking now at the relative sensor measurement, $g(\mathbf{x}_A(t)) = g(\mathbf{x}_B(t))$ implies that:

$$\begin{aligned}
&A(\mathbf{q}_A^{(1)}(t))({}^0\mathbf{p}_2 - {}^0\mathbf{p}_1) + A(\mathbf{q}_A^{(1)}(t))A(\mathbf{q}_A^{(2)-1}(t))^2\mathbf{w}_1 \\
&= A(\mathbf{q}_B^{(1)}(t))({}^0\mathbf{p}_2 - {}^0\mathbf{p}_1) + A(\mathbf{q}_B^{(1)}(t))A(\mathbf{q}_B^{(2)-1}(t))^2\mathbf{w}_1, \quad \forall t \geq 0
\end{aligned} \tag{43}$$

and since $\mathbf{q}_A^{(1)}(t) = \mathbf{q}_B^{(1)}(t)$, this implies that:

$$A(\mathbf{q}_A^{(2)-1}(t))^2\mathbf{w}_1 = A(\mathbf{q}_B^{(2)-1}(t))^2\mathbf{w}_1 \iff A(\Delta\mathbf{q}^{(2)}(t))^2\mathbf{w}_1 = {}^2\mathbf{w}_1, \tag{44}$$

where $\Delta\mathbf{q}^{(2)}(t) \triangleq \mathbf{q}_B^{(2)}(t) \otimes \mathbf{q}_A^{(2)-1}(t)$ and where we have used the fact that attitude matrices have unique inverses. This implies that either $\Delta\rho^{(2)}(t) = 0$ or $\Delta\rho^{(2)}(t)$ is parallel to ${}^2\mathbf{w}_1$ for all $t \geq 0$. We now show that $\Delta\rho^{(2)}(t)$ being parallel to ${}^2\mathbf{w}_1$ for all $t \geq 0$ is not possible. The proof is by contradiction. Assume that $\Delta\rho^{(2)}(t) \neq 0$ and $\Delta\rho^{(2)}(t)$ is parallel to ${}^2\mathbf{w}_1$ for all $t \geq 0$. This implies that $\Delta\rho^{(2)}(0)$ is parallel to ${}^2\mathbf{w}_1$, i.e. $\Delta\rho^{(2)}(0) \times {}^2\mathbf{w}_1 = 0$. Using Lemma 4 in the appendix, we can show that the state transition equation for $\Delta\rho^{(2)}(t)$ is given by:

$$\begin{aligned}
&\Delta\rho^{(2)}(t) = \\
&\Delta\rho^{(2)}(0) - \frac{\sin(\|{}^2\omega^{(2)}\|t)}{\|{}^2\omega^{(2)}\|}\Big({}^2\omega^{(2)} \times \Delta\rho^{(2)}(0)\Big) + \frac{2\sin^2\Big(\frac{1}{2}\|{}^2\omega^{(2)}\|t\Big)}{\|{}^2\omega^{(2)}\|^2}\Big({}^2\omega^{(2)} \times ({}^2\omega^{(2)} \times \Delta\rho^{(2)}(0))\Big),
\end{aligned} \tag{45}$$

where we have used the fact that the angular rates of $\mathbf{q}_A^{(2)}$ and $\mathbf{q}_B^{(2)}$ are the same when expressed in their respective frames. Since $\Delta\rho^{(2)}(0)$ is nonzero and parallel to ${}^2\mathbf{w}_1$, and since ${}^2\omega^{(2)} \times {}^2\mathbf{w}_1 \neq 0$, we know that ${}^2\omega^{(2)} \times \Delta\rho^{(2)}(0) \neq 0$. We also know that $\|{}^2\omega^{(2)}\| > 0$. From Lemma 5 in the appendix, this means that there exists a time t such that $\Delta\rho^{(2)}(t)$ is not parallel to $\Delta\rho^{(2)}(0)$, and is hence not parallel to ${}^2\mathbf{w}_1$, which is a contradiction. Hence $\Delta\rho^{(2)}(t) = 0 \;\; \forall t \geq 0$ and so $g(\mathbf{x}_A(t)) = g(\mathbf{x}_B(t)) \;\; \forall t \geq 0$ implies that $\mathbf{x}_A(0) = \mathbf{x}_B(0)$. From Definition 2 with $T = 0$ and $U = \mathcal{S}(\mathcal{N})$, this means that $I(\mathbf{x}, \mathcal{S}(\mathcal{N})) = \mathbf{x}$. Hence from Definition 3 the system described by (28) and (37) is observable at any $\mathbf{x} \in \mathcal{S}(\mathcal{N})$. □

Lemma 3 shows that, if one module has a star tracker with non-collinear stars, then the attitude of both modules can be observed using a single beacon if and only if the module with the beacon rotates and the rotation vector is not parallel to the vector to the beacon.

D. Sufficient Conditions for Observability in Fractionated Attitude Estimation

In this section we consider an arbitrary cluster of N modules with both star trackers and relative sensors. Then the cluster attitude state is:

$$\mathbf{x} \triangleq \begin{bmatrix} \mathbf{q}^{(1)} \\ \vdots \\ \mathbf{q}^{(N)} \end{bmatrix}. \tag{46}$$

We can write the cluster dynamics in the form of (21) as follows:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{q}}^{(1)} \\ \vdots \\ \dot{\mathbf{q}}^{(N)} \end{bmatrix} = f(\mathbf{x}, \mathbf{u}) = f(\mathbf{x}) = \begin{bmatrix} \frac{1}{2}\Omega({}^1\omega^{(1)})\mathbf{q}^{(1)} \\ \vdots \\ \frac{1}{2}\Omega({}^2\omega^{(2)})\mathbf{q}^{(N)} \end{bmatrix}. \tag{47}$$

The observations are given by:

$$g(\mathbf{x}) = \begin{bmatrix} A(\mathbf{q}^{(l_1)})^0\mathbf{s}_{1,1} \\ \vdots \\ A(\mathbf{q}^{(l_{n_\mathcal{T}})})^0\mathbf{s}_{n_\mathcal{T}, n_{\mathbf{b},n_\mathcal{T}}} \\ {}^{l_1}\mathbf{r}_1 \\ \vdots \\ {}^{l_{n_\mathcal{R}}}\mathbf{r}_{n_\mathcal{R}} \end{bmatrix}. \tag{48}$$

Condition 1. *A node satisfies Condition 1 if it is a star tracker node where the corresponding star tracker has a pair of star vectors ${}^0\mathbf{s}_{1,i}$ and ${}^0\mathbf{s}_{1,j}$ such that ${}^0\mathbf{s}_{1,i} \times {}^0\mathbf{s}_{1,j} \neq 0$.*

Condition 2. *A sensing arc between node i and node j satisfies Condition 2 if at least one of the following holds:*

1. *There are two or more relative sensors (k, l) on the arc with ${}^j\mathbf{w}_k \times {}^j\mathbf{w}_l \neq 0$.*
2. *There is at least one relative sensor k on the arc with ${}^j\omega^{(j)} \times {}^j\mathbf{w}_k \neq 0$*

Theorem 1. *If for every node i in the fractionated cluster $\mathcal{F}$ that does not satisfy Condition 1, there exists a path through the sensing graph from a node satisfying Condition 1 to node i, where each arc satisfies Condition 2, then the system described by (47) and (48) is observable.*

Proof: To show observability, it suffices to show that $g(\mathbf{x}_A(t)) = g(\mathbf{x}_B(t)) \; \forall t$ implies that $\mathbf{x}_A(0) = \mathbf{x}_B(0)$, for any $\mathbf{x}_A(0) \in \mathcal{S}(\mathcal{N})$. Define:

$$\mathbf{x}_A(t) = \begin{bmatrix} \mathbf{q}_A^{(1)}(t) \\ \vdots \\ \mathbf{q}_A^{(N)}(t) \end{bmatrix} \quad \mathbf{x}_B(t) = \begin{bmatrix} \mathbf{q}_B^{(1)}(t) \\ \vdots \\ \mathbf{q}_B^{(N)}(t) \end{bmatrix}. \tag{49}$$

Then, to show that $g(\mathbf{x}_A(t)) = g(\mathbf{x}_B(t))$ implies that $\mathbf{x}_A(0) = \mathbf{x}_B(0)$, it suffices to show that $g(\mathbf{x}_A(t)) = g(\mathbf{x}_B(t))$ implies that $\mathbf{q}_A^{(i)}(0) = \mathbf{q}_B^{(i)}(0)$ for all i. From Lemma 1 this is true for all i where node i satisfies Condition 1.

We now consider the case where module i does not have such a star tracker. From the assumptions of the present theorem, there exists a path from a node satisfying Condition 1 to module i, where each arc satisfies Condition 2. Let this path have length L and define p_j as the node number at location j along the path, such that $p_L = i$ and p_1 is a node satisfying Condition 1. We know show that $g(\mathbf{x}_A(0)) = g(\mathbf{x}_B(0))$ implies that $\mathbf{q}_A^{(p_j)}(0) = \mathbf{q}_B^{(p_j)}(0)$ for $j = 1, \ldots, L$. The proof is by induction. Since node p_1 satisfies Condition 1, from Lemma 1 we know that $g(\mathbf{x}_A(t)) = g(\mathbf{x}_B(t))$ implies that $\mathbf{q}_A^{(p_1)}(0) = \mathbf{x}_B^{(p_1)}(0)$. Now assume that $g(\mathbf{x}_A(t)) = g(\mathbf{x}_B(t))\ \forall t$ implies that $\mathbf{q}_A^{(p_k)}(0) = \mathbf{q}_B^{(p_k)}(0)$ for any node p_k in the path. We now show that $g(\mathbf{x}_A(t)) = g(\mathbf{x}_B(t))\ \forall t$ and $\mathbf{q}_A^{(p_k)}(0) = \mathbf{q}_B^{(p_k)}(0)$ implies that $\mathbf{q}_A^{(p_{k+1})}(0) = \mathbf{q}_B^{(p_{k+1})}(0)$. Since there is a sensing arc between node p_k and p_{k+1} there is one or more relative sensor on module p_k taking measurements of beacons on module p_{k+1}. Looking at the measurement from these sensors, $g(\mathbf{x}_A(t)) = g(\mathbf{x}_B(t))\ \forall t$ implies that:

$$\begin{aligned} &A(\mathbf{q}_A^{(p_k)}(t))({}^0\mathbf{p}_2 - {}^0\mathbf{p}_1) + A(\mathbf{q}_A^{(p_k)}(t))A(\mathbf{q}_A^{(p_{k+1})-1}(t))^{v_i}\mathbf{w}_i \\ &= A(\mathbf{q}_B^{(p_k)}(t))({}^0\mathbf{p}_2 - {}^0\mathbf{p}_1) + A(\mathbf{q}_B^{(p_k)}(t))A(\mathbf{q}_B^{(p_{k+1})-1}(t))^{v_i}\mathbf{w}_i \quad \forall t\ \forall i \text{ s.t. } u_i = p_k \text{ and } v_i = p_{k+1}, \end{aligned} \tag{50}$$

and since we have assumed that $\mathbf{q}_A^{(p_k)}(t) = \mathbf{q}_B^{(p_k)}(t)$, this implies that:

$$A(\Delta\mathbf{q}^{(p_{k+1})}(t))^{v_i}\mathbf{w}_i = {}^{v_i}\mathbf{w}_i \quad \forall t,\ \forall i \text{ s.t. } u_i = p_k \text{ and } v_i = p_{k+1}, \tag{51}$$

where $\Delta\mathbf{q}^{(p_{k+1})}(t) \triangleq \mathbf{q}_B^{(p_{k+1})}(t) \otimes \mathbf{q}_A^{(p_{k+1})-1}(t)$. We define:

$$\Delta\mathbf{q}^{(p_{k+1})}(t) = \begin{bmatrix} \Delta\rho^{(p_{k+1})}(t) \\ \Delta q_4^{(p_{k+1})}(t) \end{bmatrix}. \tag{52}$$

As with the proof of Lemmas 2 and 3, (44) implies that either $\Delta\rho^{(p_{k+1})}(t) = 0$ or $\Delta\rho^{(p_{k+1})}(t)$ is parallel to ${}^{v_i}\mathbf{w}_i$ for all i and for all $t \geq 0$. However since the relative sensor satisfies Condition 2, the latter is not possible. Hence $\Delta\rho^{(p_{k+1})}(0) = 0$, and so $\mathbf{q}_A^{(p_{k+1})}(0) = \mathbf{q}_B^{(p_{k+1})}(0)$. Then, by induction, $g(\mathbf{x}_A(0)) = g(\mathbf{x}_B(0))$ implies that $\mathbf{q}_A^{(p_j)}(0) = \mathbf{q}_B^{(p_j)}(0)$ for $j = 1, \ldots, L$ and since $p_L = i$ this means that $g(\mathbf{x}_A(0)) = g(\mathbf{x}_B(0))$ implies that $\mathbf{q}_A^{(i)}(0) = \mathbf{q}_B^{(i)}(0)$ for all i. Hence $g(\mathbf{x}_A(0)) = g(\mathbf{x}_B(0))$ implies that $\mathbf{x}_A(0) = \mathbf{x}_B(0)$ for any $\mathbf{x}_A(0) \in \mathcal{S}(\mathcal{N})$. From Definition 2 with $T = 0$ and $U = \mathcal{S}(\mathcal{N})$, this means that $I(\mathbf{x}, \mathcal{S}(\mathcal{N})) = \mathbf{x}$. Hence from Definition 3 the system described by (47) and (48) is observable at any $\mathbf{x} \in \mathcal{S}(\mathcal{N})$. □

Theorem 1 shows that the attitude of every module in the cluster can be observed if every module has either a star tracker with non-collinear stars, or there is a path through the sensing network from a module with a star tracker to the module without a star tracker, and each of the relative measurements along the path has either multiple non-collinear beacons or a single beacon that is not parallel to the rotation vector of the target module.

V. Conclusion

In this paper we have presented sufficient conditions for observability of the attitude of a fractioned spacecraft system. We have shown that the attitude of every module in the cluster can be observed if every module has either a star tracker with non-collinear stars, or there is a path through the sensing network from a module with a star tracker to the module without a star tracker, and each of the relative measurements along the path has either multiple non-collinear beacons or a single beacon that is not parallel to the rotation vector of the target module. Our ongoing work uses the observability results presented here to develop practical nonlinear estimation techniques and to determine empirically how the attitude estimation capabilities of the fractionated system depend on the capabilities of the individual modules.

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VII. Appendix

Lemma 4. *Consider two quaternions $\mathbf{q}_1$ and $\mathbf{q}_2$ defining rotations between a reference frame $\mathbb{F}^{(0)}$ and frames $\mathbb{F}^{(1)}$ and $\mathbb{F}^{(2)}$ respectively. Frame $\mathbb{F}^{(1)}$ has rotation rate ${}^1\omega_1$ and frame $\mathbb{F}^{(2)}$ has rotation rate ${}^2\omega_2$. Define the difference quaternion as $\Delta\mathbf{q} = \mathbf{q}_2 \otimes \mathbf{q}_1^{-1}$. Further define $\Delta\rho$ and Δq_4 such that:*

$$\Delta\mathbf{q} = \begin{bmatrix} \Delta\rho \\ \Delta q_4 \end{bmatrix}. \tag{53}$$

Then if ${}^1\omega_1 = {}^2\omega_2 = \omega$, the state transition function for $\Delta\rho$ is given by:

$$\Delta\rho(t) = \Delta\rho(0) - \frac{\sin(\|\omega\|t)}{\|\omega\|}\Big(\omega \times \Delta\rho(0)\Big) + \frac{2\sin^2\Big(\frac{1}{2}\|\omega\|t\Big)}{\|\omega\|^2}\Big(\omega \times (\omega \times \Delta\rho(0))\Big). \tag{54}$$

Proof: Ref. 23 shows that, for constant ${}^1\omega_1$ and ${}^2\omega_2$, the state transition matrix for the difference quaternion $\Delta\mathbf{q}$ is given by:

$$\Delta\mathbf{q}(t) = \bar{\Omega}({}^2\omega_2)\bar{\Gamma}({}^1\omega_1)\Delta\mathbf{q}(0), \tag{55}$$

where:

$$\bar{\Omega}({}^2\omega_2) \triangleq \begin{bmatrix} \cos\Big(\frac{1}{2}\|{}^2\omega_2\|t\Big)I_3 - [\psi(t)\times] & \psi(t) \\ -\psi^T(t) & \cos\Big(\frac{1}{2}\|{}^2\omega_2\|t\Big) \end{bmatrix} \tag{56}$$

$$\bar{\Gamma}({}^1\omega_1) \triangleq \begin{bmatrix} \cos\Big(\frac{1}{2}\|{}^1\omega_1\|t\Big)I_3 - [\zeta(t)\times] & -\zeta(t) \\ \zeta^T(t) & \cos\Big(\frac{1}{2}\|{}^1\omega_1\|t\Big) \end{bmatrix} \tag{57}$$

and:

$$\psi(t) \triangleq \frac{\sin\Big(\frac{1}{2}\|{}^2\omega_2\|t\Big){}^2\omega_2}{\|{}^2\omega_2\|} \tag{58}$$

$$\zeta(t) \triangleq \frac{\sin\Big(\frac{1}{2}\|{}^1\omega_1\|t\Big){}^1\omega_1}{\|{}^1\omega_1\|}. \tag{59}$$

We can write:

$$\Delta\mathbf{q}(t) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\Delta\mathbf{q}(0), \tag{60}$$

where:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos\Big(\frac{1}{2}\|\omega\|t\Big)I_3 - [\psi(t)\times] & \psi(t) \\ -\psi^T(t) & \cos\Big(\frac{1}{2}\|\omega\|t\Big) \end{bmatrix} \begin{bmatrix} \cos\Big(\frac{1}{2}\|\omega\|t\Big)I_3 - [\zeta(t)\times] & -\zeta(t) \\ \zeta^T(t) & \cos\Big(\frac{1}{2}\|\omega\|t\Big) \end{bmatrix}. \tag{61}$$

Using the fact that ${}^1\omega_1 = {}^2\omega_2 = \omega$ and multiplying out the matrices to find B, we get:

$$B = -\zeta(t)\cos\Big(\frac{1}{2}\|\omega\|t\Big) + [\psi(t)\times]\zeta(t) + \psi(t)\cos\Big(\frac{1}{2}\|\omega\|t\Big) \tag{62}$$

$$= -\psi(t)\cos\Big(\frac{1}{2}\|\omega\|t\Big) + [\psi(t)\times]\psi(t) + \psi(t)\cos\Big(\frac{1}{2}\|\omega\|t\Big) = 0. \tag{63}$$

Since $B = 0$, we can write $\Delta\rho(t) = A\Delta\rho(0)$, and multiplying to find A it can be shown that:

$$\begin{aligned} A &= \cos^2\Big(\frac{1}{2}\|\omega\|t\Big)I_3 - 2[\psi(t)\times]\cos\Big(\frac{1}{2}\|\omega\|t\Big) + [\psi(t)\times][\psi(t)\times] + \psi(t)\psi(t)^T \\ &= I_3 - \frac{\sin(\|\omega\|t)}{\|\omega\|}[\omega\times] + \frac{2\sin^2(\frac{1}{2}\|\omega\|t)}{\|\omega\|^2}[\omega\times][\omega\times], \end{aligned} \tag{64}$$

where we have used the identity $\mathbf{a}\mathbf{a}^T \equiv \|\mathbf{a}\|^2 I_3 + [\mathbf{a}\times][\mathbf{a}\times]$ for a vector $\mathbf{a} \in \Re^3$. Substituting (64) into $\Delta\rho(t) = A\Delta\rho(0)$ we obtain:

$$\Delta\rho(t) = \Delta\rho(0) - \frac{\sin(\|\omega\|t)}{\|\omega\|}\Big(\omega \times \Delta\rho(0)\Big) + \frac{2\sin^2\Big(\frac{1}{2}\|\omega\|t\Big)}{\|\omega\|^2}\Big(\omega \times (\omega \times \Delta\rho(0))\Big). \tag{65}$$

□

Lemma 5. *Let:*

$$\Delta\rho(t) = \Delta\rho(0) - \frac{\sin(\|\omega\|t)}{\|\omega\|}\Big(\omega \times \Delta\rho(0)\Big) + \frac{2\sin^2\Big(\frac{1}{2}\|\omega\|t\Big)}{\|\omega\|^2}\Big(\omega \times (\omega \times \Delta\rho(0))\Big). \tag{66}$$

If $\Delta\rho(0) \neq 0$, ω is not parallel to $\Delta\rho(0)$ and $\omega \neq 0$, then there exists a time $t \geq 0$ such that $\Delta\rho(t)$ is not parallel to $\Delta\rho(0)$.

Proof: Define:

$$\begin{aligned} \mathbf{u} &\triangleq \omega \times \Delta\rho(0) \quad \mathbf{v} \triangleq \omega \times (\omega \times \Delta\rho(0)) \\ \mathbf{w} &\triangleq -\frac{\sin(\|\omega\|t)}{\|\omega\|}\mathbf{u} + \frac{2\sin^2\Big(\frac{1}{2}\|\omega\|t\Big)}{\|\omega\|^2}\mathbf{v} \end{aligned} \tag{67}$$

Then $\Delta\rho(t)$ is parallel to $\Delta\rho(0)$ only if $\mathbf{w}$ is zero or $\mathbf{w}$ is parallel to $\Delta\rho(0)$. Since $\omega \times \Delta\rho(0) \neq 0$ we know that $\mathbf{u} \neq 0$, and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we know that $\mathbf{w} \neq 0$ for all t such that $0 < t < \frac{\pi}{\|\omega\|}$. We now show that there exists a t such that $\mathbf{w}$ is not parallel to $\Delta\rho(0)$. Let us first assume that there exists a time t^* such that $\mathbf{w}$ is parallel to $\Delta\rho(0)$, that is:

$$k\Delta\rho(0) = -\frac{\sin(\|\omega\|t^*)}{\|\omega\|}\mathbf{u} + \frac{2\sin^2\Big(\frac{1}{2}\|\omega\|t^*\Big)}{\|\omega\|^2}\mathbf{v}, \tag{68}$$

where k is a nonzero scalar. Rewriting (68) we obtain:

$$k\Delta\rho(0) = a\mathbf{u} + b\mathbf{v} \tag{69}$$

$$a = -\frac{\sin(\|\omega\|t^*)}{\|\omega\|} \quad b = \frac{2\sin^2\Big(\frac{1}{2}\|\omega\|t^*\Big)}{\|\omega\|^2}. \tag{70}$$

From the definition of $\mathbf{u}$ we know that the vectors $\mathbf{u}$ and $\Delta\rho(0)$ are not parallel, and we also know that $\mathbf{u}$ and $\mathbf{v}$ are not parallel. If $\mathbf{v}$ is parallel to $\Delta\rho(0)$ then $\Delta\rho(t)$ is not parallel to $\Delta\rho(0)$ for all $0 < t < \frac{\pi}{\|\omega\|}$ since $\Delta\rho(t)$ is the sum of two vectors parallel to $\Delta\rho(0)$ and one non-zero vector not parallel to $\Delta\rho(0)$. If $\mathbf{v}$ is not parallel to $\Delta\rho(0)$ then the matrix formed by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\Delta\rho(0)$ is invertible, meaning that (68) has at most one solution for (a, b, k) and hence (68) has at most one solution for t^* on $0 < t < \frac{\pi}{2\|\omega\|}$. Hence any t such that $0 < t < \frac{\pi}{2\|\omega\|}$ and $t \neq t^*$ does not satisfy (68), from which we conclude that there exists a t such that $\mathbf{w}$ is not parallel to $\Delta\rho(0)$. Since $\mathbf{w} \neq 0$ for all t such that $0 < t < \frac{\pi}{\|\omega\|}$, this means there exists a t such that $\Delta\rho(t)$ is not parallel to $\Delta\rho(0)$. □

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