Lossless Convexification of a Class of Optimal Control Problems with Non-Convex Control Constraints

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Abstract

We consider a class of finite time horizon optimal control problems for continuous time linear systems with a convex cost, convex state constraints and non-convex control constraints. We propose a convex relaxation of the non-convex control constraints, and prove that the optimal solution of the relaxed problem is also an optimal solution for the original problem, which is referred to as the lossless convexification of the optimal control problem. The lossless convexification enables the use of interior point methods of convex optimization to obtain globally optimal solutions of the original non-convex optimal control problem. The solution approach is demonstrated on a number of planetary soft landing optimal control problems.

1 Introduction

This paper analyzes a class of finite time horizon optimal control problems with a convex cost, convex state and non-convex control constraints. There are a variety of optimal control problems that fall into this class. One interesting example is the planetary landing problem [20,15,1,5], also known as the soft landing problem. In planetary landing, an autonomous spacecraft lands on the surface of a planet by using thrusters, which produce a force vector that has both an upper and a nonzero lower bound on its magnitude. The nonzero lower bound exists because the thrusters cannot operate reliably under this bound. This constraint on the magnitude makes the set of feasible controls non-convex. Solution of planetary soft landing problems are needed for manned and robotic missions to Mars, the Moon, and asteroids.

The optimal control problems considered in this paper have convex cost, linear system dynamics and convex state constraints. Hence the control constraints are the single source of non-convexity. In this paper, we introduce a *convex relaxation* of the control constraints, which proceeds as follows. The non-convex set of feasible control inputs is replaced by a convex set by introducing a slack variable. It is first shown that the optimal solution of the relaxed problem also defines a feasible solution of the original one when the optimal state trajectory of the relaxed problem is strictly in the interior of the set of feasible states. It is then shown that under further assumptions, the optimal trajectories of the relaxed problem are also optimal for the original problem. Hence a *lossless convexification* can be achieved. We use lossless convexification to refer to obtaining a convex relaxation of an optimization problem, where an optimal solution of the relaxed problem also defines an optimal solution for the original non-convex problem. These results are then extended to cases where some portions of the optimal state trajectories for the relaxed problem lie on the boundary.

In general the existence of the state and control constraints means that closed form solutions to the instances of the relaxed control problem do not exist. We can overcome this difficulty by considering direct numerical methods to compute the optimal solutions, where the original infinite-dimensional control problem is approximated by a finite-dimensional parameter optimization problem [14,4,9,25]. Since the relaxed control problem is convex, the resulting parameter optimization problem is also convex. A convex optimization problem, under mild computability and regularity assumptions, is solvable to global optimality in polynomial time with an a priori known upper bound on the number of mathematical operations needed [21,6,24]. Hence the convexification leads to a computationally tractable solution method, which is critically important for autonomous real-time control applications. By contrast, approaches that solve the original non-convex problem using nonlinear programming techniques can only guarantee convergence to a local optimum, and can fail to find a feasible solution unless a feasible initial guess is provided by the user. Examples of such approaches include[13,12],

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which use a nonsmooth Newton method to find the local optimum, and [7,18], which use Sequential Quadratic Programming[22]. For impulsive optimal control problems, [26] uses a nonlinear optimization approach combined with an approach for exploring different local optima until the global optimum is found, however this approach does not provide the convergence guarantees of our convex optimization approach. Finally, the convexification enables the use of receding horizon model predictive control to obtain a robust feedback control action [11,19].

The following is a partial list of notation used: $\mathbb{R}$ is the set of real numbers; a condition is said to hold almost everywhere in the interval $[a,b]$, a.e. $[a,b]$, if the set of points in $[a,b]$ where this condition fails to hold is in a set of measure zero; $\mathbb{R}^n$ is the n dimensional real vector space; $\emptyset$ denotes the empty set; $\|v\|$ is the 2-norm of the vector v; $\mathbf{0}$ is matrix of zeros; I is the identity matrix; e_i is a column vector with its ith entry 1 and other entries zero; $(v_1, v_2, ..., v_m)$ represents a vector obtained by augmenting vectors $v_1, \ldots, v_m$ such that: $(v_1, v_2, ..., v_m) := \left[v_1^T \; v_2^T \; \ldots \; v_m^T\right]^T$; $\mathrm{diag}(A_1, ..., A_n)$, is the matrix with diagonal blocks formed by matrices $A_1, ..., A_n$ and off-diagonals are zero matrices of appropriate size; $\partial\mathcal{S}$ denotes the set of boundary points, $\mathrm{int}\mathcal{S}$ denotes the interior, and $\mathrm{cl}\mathcal{S}$ denotes the closure of the set $\mathcal{S}$; $\mathbb{R}_+$ denotes the set of nonnegative real numbers; $\mathcal{U}^\dagger$ is defined for $\mathcal{U} \subset \mathbb{R}^n$ as $\mathcal{U}^\dagger := \left\{v \in \mathbb{R}^n : \; \exists\, c \in \mathbb{R} \text{ s.t. } v^T u = c \;\; \forall u \in \mathcal{U}\right\}$.

2 Problem Formulation

This section introduces the optimal control problem studied in this paper. The assumptions introduced here are used throughout the paper.

Original Optimal Control Problem (OCP)

$$\begin{aligned} \min_{\omega_t, \omega_x, u} & h_0(t_0, t_f, x(t_0), x(t_f)) + k \int_{t_0}^{t_f} g_0(u(t)) dt \;\text{ s.t.} \\ & \dot{x}(t) = A(t)x(t) + B(t)u(t) + E(t)w(t) \\ & x(t) \in \mathcal{X} \quad \text{and} \quad u(t) \in \mathcal{U} \quad \text{a.e. } [t_0, t_f] \\ & (t_0, t_f, x(t_0), x(t_f)) \in \mathcal{E} \end{aligned} \tag{1}$$

where t is the time, t_0 is the initial time, t_f is the finite final time, $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^p$ is a known exogenous input, $h_0 : \mathbb{R}^{2n+2} \to \mathbb{R}$ is a convex function describing the cost on the end states, $g_0 : \mathbb{R}^m \to \mathbb{R}$ is a convex function describing the integral cost on the control input, $k \geq 0$ is a scalar, $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, $B : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$, and $E : \mathbb{R}_+ \to \mathbb{R}^{n\times p}$ are piecewise analytic functions of time, $\mathcal{X} \subseteq \mathbb{R}^n$ is the set of feasible states, $\mathcal{U} \in \mathbb{R}^m$ is the set of feasible control inputs, and $\mathcal{E} \subset \mathbb{R}^{2n+2}$ is the set of feasible *boundary conditions*

$$\mathcal{E} = \left\{z \in \mathbb{R}^{2n+2} \colon z = a_b + L_b \omega_b, \; \omega_b = (\omega_t, \omega_x) \in \mathbb{R}^{q_e}\right\} \tag{2}$$

where $\omega_t \in \mathbb{R}^{q_{t,e}}$ and $\omega_x \in \mathbb{R}^{q_{x,e}}$ are the decision variables describing the degrees of freedom (DOFs) in the initial and final states and times, $q_e = q_{t,e} + q_{x,e}$ is the number of DOFs, $a_b = (a_t, a_x)$ with $a_t \in \mathbb{R}^2$ and $a_x \in \mathbb{R}^{2n}$ are prescribed boundary times and states, respectively, and $L_b = \mathrm{diag}(L_t, L_x)$ with $L_x \in \mathbb{R}^{2n\times q_{x,e}}$ and $L_t \in \mathbb{R}^{2\times q_{t,e}}$ satisfy $L_x^T L_x = I$ and $L_t^T L_t = I$. Here $\mathcal{X}$ is a convex set and $\mathcal{U}$ is a non-convex set that satisfies

$$\mathcal{U} = \mathcal{U}_1 \setminus \mathcal{U}_2, \quad \mathcal{U}_2 = \bigcap_{i=1}^{q} \mathcal{U}_{2,i} \subset \mathcal{U}_1 \;\text{ where} \tag{3}$$

$\mathcal{U}_1$ and $\mathcal{U}_2$ are, respectively, compact convex and open convex sets with

$$\mathcal{U}_{2,i} = \{u \in \mathbb{R}^m : \; g_i(u) < 1\}, \quad i = 1, ..., q, \tag{4}$$

where g_i, $i = 1, ..., q$, are convex functions that are bounded on $\mathcal{U}_1$, that is, there exists some $\bar{g} \in \mathbb{R}$ such that $g_i(u) \leq \bar{g}$, $\forall u \in \mathcal{U}_1$, $i = 1, ..., q$. Note that $\mathcal{U}_2 \cap \partial\mathcal{U}_1$ is empty. This follows from the fact that $\mathcal{U}_2 \subset \mathcal{U}_1$ and $\mathcal{U}_2 \cap \partial\mathcal{U}_2$ is empty. The main difficulty in the convexification of the OCP is the non-convex control constraints defined by the set $\mathcal{U}$ (see Figure 1), which is the subject of this paper.

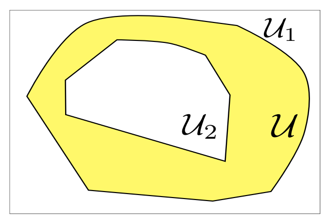

Fig. 1. *Non-convex set of feasible control inputs, $\mathcal{U}$, shaded.*

3 Main Technical Result

This section presents Theorem 2, which is the main theoretical result of this paper and it is based on the following convex relaxation of the OCP given by (1):

Relaxed Optimal Control Problem (RCP)

$$\begin{aligned} \min_{\omega_t, \omega_x, \omega_\xi, u, \sigma} & h_0(t_0, t_f, x(t_0), x(t_f)) + k\,\xi(t_f) \;\text{ s.t.} \\ & \dot{x}(t) = A(t)x(t) + B(t)u(t) + E(t)w(t), \\ & \dot{\xi}(t) = \sigma(t) \\ & x(t) \in \mathcal{X} \;\text{ and } \; (u(t), \sigma(t)) \in \mathcal{V} \quad \text{a.e. } [t_0, t_f] \\ & (t_0, t_f, x(t_0), x(t_f), \xi(t_0), \xi(t_f)) \in \tilde{\mathcal{E}} \end{aligned} \tag{5}$$

where $\sigma(t) \in I\!R$ is a slack control variable, $\xi(t) \in I\!R$ is a slack state variable with $\omega_\xi \in I\!R$ describing its DOF at t_f, $\tilde{n} = 2n+4$,

$$\tilde{\mathcal{E}} = \left\{ \tilde{z} \in I\!R^{\tilde{n}} : \; \tilde{z} = \tilde{a} + T\tilde{\omega}, \; \tilde{\omega} = (\omega_t, \omega_x, \omega_\xi) \in I\!R^{q_e+1} \right\},$$

$$\tilde{a} = (a_t, a_x, \mathbf{0}), \; T = \operatorname{diag}(L_t, L_x, \mathrm{e}_2),$$

$$\mathcal{V} = \left\{ (u,\sigma) \in I\!R^{m+1} : \sigma \geq 1 \;\text{ and }\; u \in \mathcal{U}_1 \cap \mathcal{V}_2(\sigma) \right\} \quad (6)$$

$$\text{with} \qquad \mathcal{V}_2(\sigma) = \bigcap_{i=i_o}^{q} \mathcal{V}_{2,i}(\sigma) \quad \text{where} \qquad (7)$$

$$\mathcal{V}_{2,i}(s) := \{ u \in I\!R^m : g_i(u) \leq s \}, \; i_o = \begin{cases} 0 \text{ for } k > 0 \\ 1 \text{ for } k = 0 \end{cases}.$$

Figure 2 illustrates the set of feasible control inputs of RCP, $\mathcal{V}$, versus $\mathcal{U}$ of the OCP on the planetary soft landing example described in Section 5. In this case, we have $\mathcal{U} = \left\{ u \in I\!R^2 : 1 \leq \|u\| \leq \rho \right\}$ and

$$\mathcal{V} = \left\{ (u,\sigma) \in I\!R^3 : \sigma \geq 1, \; \|u\| \leq \min(\rho,\sigma) \right\}.$$

Clearly $\mathcal{V}$ is in a higher dimensional space and $\mathcal{U} \subset \mathcal{V}$. The set $\mathcal{V}$ also contains control inputs that are not feasible for the OCP. Hence it is not trivial to establish that an optimal solution of the RCP will also be feasible for the OCP. We will prove that the minimum fuel optimal controls of the RCP will satisfy that $1 \leq \|u\| = \sigma \leq \rho$, hence will be feasible for the OCP.

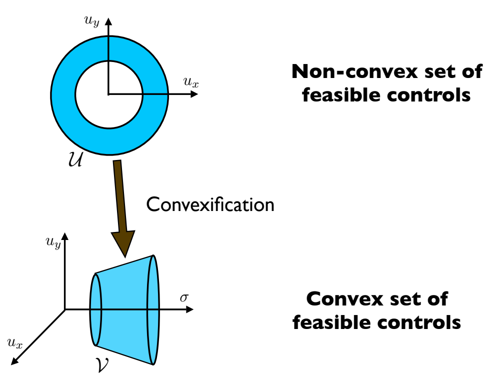

Fig. 2. *Convexification of the Control Magnitude Constraint for Soft Landing. The annulus represents the actual non-convex control constraints in (u_x, u_y) space, which is lifted to a convex cone in (u_x, u_y, σ) space for the RCP (5).*

Definition 1 *The sets of all feasible solutions of the OCP and the RCP, (1) and (5), are denoted by $\mathcal{F}_O$ and $\mathcal{F}_R$ respectively. $(t_0, t_f, x, u) \in \mathcal{F}_O$ and $(t_0, t_f, x, \xi, u, \sigma) \in \mathcal{F}_R$ if they satisfy the differential equations and the state and the control constraints of the OCP and the RCP respectively* a.e. $[t_0, t_f]$. *$\mathcal{F}_O^*$ and $\mathcal{F}_R^*$ represent the respective sets of optimal solutions, with optimal costs $\mathbf{J}_O^*$ and $\mathbf{J}_R^*$. Unless otherwise stated,* a solution *of OCP or RCP refers to* a globally optimal solution.

The results of this paper will establish useful relationships between the solutions of the OCP given by (1) and the RCP given by (5).

Theorem 1 *Then following relationships hold for the OCP and RCP given by (1) and (5): (i) If $(t_0, t_f, x, u) \in \mathcal{F}_O$, then $\exists\, (\xi, \sigma)$ s.t. $(t_0, t_f, x, \xi, u, \sigma) \in \mathcal{F}_R$.*
(ii) If $(t_0, t_f, x, \xi, u, \sigma) \in \mathcal{F}_R$ s.t. $u(t) \in \mathcal{U}$ a.e. $[t_0, t_f]$, *then $(t_0, t_f, x, u) \in \mathcal{F}_O$.*

Proof: Proof of (i): Let $\sigma(t) = \max_{i=i_o,\ldots,q} g_i(u(t))$ and $\dot{\xi}(t) = \sigma(t)$ with $\xi(t_0) = 0$, which implies that $\sigma(t) \geq 1$. Since $\mathcal{U} \subset \mathcal{V}$, we have $(u(t), \sigma(t)) \in \mathcal{V}$, also $x(t) \in \mathcal{X}$ a.e. $[t_0, t_f]$ and $(x(t), u(t))$ satisfy the dynamics in (1). Consequently $(t_0, t_f, x, \xi, u, \sigma) \in \mathcal{F}_R$.

Proof of (ii): Having $u(t) \in \mathcal{U}$, $x(t) \in \mathcal{X}$ a.e. $[t_0, t_f]$ and $(x(t), u(t))$ satisfying the dynamics in (5) imply that $(t_0, t_f, x, u) \in \mathcal{F}_O$. ■

The two conditions below and the following lemmas are instrumental in our upcoming results.

Condition 1 *The pair $\{A(\cdot), B(\cdot)\}$ is* controllable *and the set of feasible controls $\mathcal{U}$ satisfies $\mathcal{U}_2^\dagger = \{\mathbf{0}\}$.*

Condition 2 *$(t_0, t_f, x, \xi, u, \sigma) \in \mathcal{F}_R^*$ and*

$$\left(-k\sigma(t_0) - \frac{\partial h_0}{\partial t_0}, \; k\sigma(t_f) - \frac{\partial h_0}{\partial t_f}, \; \frac{\partial h_0}{\partial x(t_0)}, \; \frac{\partial h_0}{\partial x(t_f)} \right) \quad (8)$$

is not orthogonal to $\mathcal{E}$, where $\mathcal{E}$ is given by (2).

Lemma 1 *Consider the system: $\dot{\lambda}(t) = -A(t)^T \lambda(t)$, $y(t) = B(t)^T \lambda(t)$, defined for $t \in [t_0, t_f]$ where $\{A(\cdot), B(\cdot)\}$ a controllable pair. If $y(t) = 0 \;\; \forall t \in [t_1, t_2] \subseteq [t_0, t_f]$ then $\lambda(t) = 0 \;\; \forall t \in [t_0, t_f]$.*

Proof: It suffices to show that $\lambda(t) = 0 \;\; \forall t \in [t_0, t_1] \cup [t_2, t_f]$. Since $\{A(\cdot), B(\cdot)\}$ is controllable, this system is observable [16]. This, combined with $\lambda(t_1) = 0$ satisfies $y(t) = 0 \;\; \forall t \in [t_1, t_2]$, implies that $\lambda(t) = 0 \;\; \forall t \in [t_1, t_2]$, and hence $\lambda(t) = 0 \;\; \forall t > t_2$. Since $\lambda(t) = \Phi(t, t_0)\lambda(t_0)$, where $\Phi(t, t_0)$ is invertible for $t \geq t_0$, and since $\lambda(t_1) = 0$, we have $\lambda(t_0) = 0$. As a result $\lambda(t) = 0 \;\; \forall t \in [t_0, t_1]$. ■

Lemma 2 *Consider any hyperplane $\mathcal{E} = \{z : z = a + T\omega, \; \omega \in I\!R^m\}$ where $a \in I\!R^n$ and $\omega \in I\!R^m$ with $m < n$. We say v is orthogonal to $\mathcal{E}$ if $v^T(z_1 - z_2) = 0$ for any of z_1 and z_2 in $\mathcal{E}$. Then v is orthogonal to $\mathcal{E}$ if and only if $v \in \mathcal{E}^\dagger$, which is equivalent to $T^T v = \mathbf{0}$.*

Proof: The proof follows directly from the discussion in Section 2.2.1 of [6]. ■

Remark 1 *A discussion of Condition 2:* Lemma 2 implies that a vector $(\phi_0, \phi_f, \psi_0, \psi_f)$ is not orthogonal to $\mathcal{E}$

if and only if $(\phi_0, \phi_f, \psi_0, \psi_f) \notin \mathcal{E}^\dagger$, which is also equivalent to one of the following conditions

$$L_t^T \begin{bmatrix} \phi_0 \\ \phi_f \end{bmatrix} \neq 0 \quad \text{or} \quad L_x^T \begin{bmatrix} \psi_0 \\ \psi_f \end{bmatrix} \neq 0. \tag{9}$$

In many cases it is straightforward to verify this condition. For example, suppose that t_0 is fixed and t_f is a free variable, which implies that $L_t = [\,0\;\;1\,]^T$. If $\partial h_0/\partial t_f = 0$ and $k>0$ (which implies that $k\sigma(t_f) > 0$), then, by letting $\phi_f = k\sigma(t_f) - \partial h_0/\partial t_f$, we can show that the first condition in (9) is satisfied. Another example is when h_0 is a strictly convex function of $x(t_f)$, which satisfies that $\partial h_0/\partial x(t_f) = Cx(t_f)$, and $x(t_0)$ is specified. Suppose that $Cx(t_f)$ is the free part of the final state. Hence $L_x = \begin{bmatrix}\mathbf{0} & C^T\end{bmatrix}^T$. Now, if $Cx(t_f) \neq 0$ for any solution of the RCP then the second inequality in (9) holds.

Remark 2 *A discussion of* $\mathcal{U}_2^\dagger = \{\mathbf{0}\}$*:* This condition can be straightforward to verify in many cases. For example, suppose that, for any $\hat{v} \in I\!R^m$ such that $\|\hat{v}\| = 1$, there exists some $a<b$ such that $V(\hat{v}) := \{u \in I\!R^m : u = \lambda\hat{v},\; a \leq \lambda \leq b\} \subset \mathcal{U}_2$. Note that every finite volume set $\mathcal{U}_2 \subset I\!R^m$ satisfies this property. In this case $\mathcal{U}_2^\dagger = \{\mathbf{0}\}$.

Our next theorem presents a fundamental result that establishes conditions under which $u(t) \in \mathcal{U}$ a.e. $[t_0, t_f]$ for the solution of the RCP.

Theorem 2 *Suppose that Condition 1 holds for the OCP (1). If $(t_0, t_f, x, \xi, u, \sigma)$ satisfies Condition 2 and*

$$x(t) \in \operatorname{int}\mathcal{X} \qquad \forall\, t \in [t_0, t_f], \tag{10}$$

then $(t_0, t_f, x, u) \in \mathcal{F}_O$.

Proof: This result will be proven by showing that an optimal control input of the RCP satisfies $u(t) \in \mathcal{U}_1$ and/or $u(t) \in \mathcal{V}_2(\sigma(t))$ $\forall t \in [t_0, t_f]$, hence it is also feasible for the OCP. This will be done by using the Maximum Principle of the optimal control theory.

Let $(t_0, t_f, x, \xi, u, \sigma) \in \mathcal{F}_R^*$. Since the condition (10) holds, by using the Maximum Principle (see Section V.3 of [2] or Chapter 1 of [23]), there exist a constant $\alpha \leq 0$ and absolutely continuous vector functions λ and η, the *co-state vectors*, such that the following hold:

[i] *Nonzero Co-states*

$$\mu(t) := \Big(\alpha,\; \lambda(t),\; \eta(t)\Big) \neq 0, \qquad \forall\, t \in [t_0, t_f]. \tag{11}$$

[ii] *Co-state Dynamics*

$$\begin{aligned} \dot{\lambda}(t) &= -A(t)^T\lambda(t) \\ \dot{\eta}(t) &= 0 \end{aligned} \qquad \text{a.e. } t \in [t_0, t_f] \tag{12}$$

[iii] *Pointwise Maximum Principle*

$$H\,(\phi(t)) = M(t, x(t), \xi(t), \mu(t)) \;\; \text{a.e.}\, t \in [t_0, t_f] \tag{13}$$

where H is the *Hamiltonian* defined by

$$H(\phi) := \eta\sigma + \lambda^T[A(t)x + B(t)u + E(t)w(t)] \tag{14}$$

$$\text{and } M(t, x, \xi, \mu) := \max_{(u,\sigma)\in\mathcal{V}} H(t, x, \xi, u, \sigma, \mu), \tag{15}$$

$$\phi(t) := (t, x(t), \xi(t), u(t), \sigma(t), \mu(t))\,. \tag{16}$$

[iv] *Transversality Condition*

$$L_x^T G_x = \mathbf{0}, \;\; L_t^T G_t = \mathbf{0}, \;\; \eta(t_f) = -\alpha k, \tag{17}$$

where

$$G_x = \left(-\lambda(t_0) - \alpha\frac{\partial h_0}{\partial x(t_0)}, \lambda(t_f) - \alpha\frac{\partial h_0}{\partial x(t_f)}\right),$$
$$G_t = \left(H(\phi(t_0)) - \alpha\frac{\partial h_0}{\partial t_0}, -H(\phi(t_f)) - \alpha\frac{\partial h_0}{\partial t_f}\right).$$

The conditions of optimality from (i)-(iii) follow directly from the statement of the Maximum Principle. But the transversality condition in (iv) requires further explanation. It implies that (see Sec. V.3 of [2]), for a solution in $\mathcal{F}_R^*$, the vector $\psi(\phi(t_0), \phi(t_f)) := (G_t, G_x, \eta(t_0), -\eta(t_f) - \alpha k)$ must be orthogonal to the manifold defined by the set of feasible boundary conditions $\tilde{\mathcal{E}}$ at $(\phi(t_0), \phi(t_f))$. $v \in \tilde{\mathcal{E}}$ if and only if $v = \tilde{a} + T\omega$ where $\omega \in I\!R^{q_e+1}$. Hence, by using Lemma 2, $\psi(\phi(t_0), \phi(t_f))$ defined above is orthogonal to $\tilde{\mathcal{E}}$ if and only if $T^T\psi(\phi(t_0), \phi(t_f)) = 0$, which is equivalent to equation (17). Next we claim that

$$y(t) := B(t)^T\lambda(t) \neq 0 \qquad \text{a.e. } t \in [t_0, t_f]. \tag{18}$$

Since A and B are piecewise analytic functions of time, λ and y are piecewise analytic over $[t_0, t_f]$ (by using Theorem 3 on p.213 of [8] and the product of analytic functions is also analytic). Hence y has a finite number of intervals, $[t_1, t_2] \subset [t_0, t_f]$, in which either $y(t) = 0$ or the set of t for which $y(t) = 0$ is countable (see Proposition 4.1 on p.41 of [8]). We now show that none of these intervals has $y(t) = 0$. Suppose that $y(t) = 0$ $\forall\, t \in [t_1, t_2] \subset [t_0, t_f]$. Since η is absolutely continuous with $\dot{\eta} = 0$, $\eta(t_f) = -\alpha k$ from (17) implies that $\eta(t) = -\alpha k$ $\forall t \in [t_0, t_f]$. Since the pair $\{A(\cdot), B(\cdot)\}$ is controllable, it follows from Lemma 1 and $\eta = -\alpha k$ that $\lambda(t) = 0$ $\forall\, t \in [t_0, t_f]$. Hence the transversality condition (17) implies that:

$$\alpha L_x^T \begin{bmatrix} \dfrac{\partial h_0}{\partial x(t_0)} \\[2ex] \dfrac{\partial h_0}{\partial x(t_f)} \end{bmatrix} = 0 \;\text{ and }\; \alpha L_t^T \begin{bmatrix} -k\sigma(t_0) - \dfrac{\partial h_0}{\partial t_0} \\[2ex] k\sigma(t_f) - \dfrac{\partial h_0}{\partial t_f} \end{bmatrix} = 0. \tag{19}$$

Since $\mathcal{E}$ is a hyperplane, Lemma 2 implies that $\mathcal{E}^\dagger = \{(s,z) : L_t^T s = 0,\ L_x^T z = 0\}$, which, with (19), concludes that $\alpha = 0$ when Condition 2 holds. So we have: $(\alpha, \lambda(t), \eta(t)) = \mathbf{0}$ a.e. $t \in [t_0, t_f]$. Since the solution must also satisfy (11), this is a contradiction, which resulted from assuming the existence of an interval $[t_1, t_2]$ such that $y(t) = 0\ \forall t \in [t_1, t_2]$. Hence no such interval exists, and the set of times for which $y(t) = 0$ is countable for every interval. This, combined with the existence of a finite number of such intervals on $t \in [t_0, t_f]$, implies that $\{t : y(t) = 0,\ t \in [t_0, t_f]\}$ is a countable set (see Proposition 4.1 on p.41 of [8]), with measure zero. Consequently $y(t) \neq 0$ a.e. $t \in [t_0, t_f]$.

Next we will prove that the optimal controls can only be on the boundary of the feasible set. The pointwise maximum principle (13) implies that, for a.e. $t \in [t_0, t_f]$,

$$y(t)^T u(t) - \alpha k \sigma(t) = \max_{(z,v) \in \mathcal{V}} \left(y(t)^T z - \alpha k v \right).$$

Hence, the optimal control pair $\{u(t), \sigma(t)\}$ must satisfy:

$$u(t) = \underset{z \in \mathcal{Z}(\sigma(t))}{\operatorname{argmax}}\ y(t)^T z, \tag{20}$$

where $\mathcal{Z}(\sigma(t)) := \mathcal{U}_1 \cap \mathcal{V}_2(\sigma(t))$. Since $\mathcal{U}_2 \subset \mathcal{Z}(\sigma(t))$ for $\sigma(t) \geq 1$ and $\mathcal{U}_2^\dagger = \{0\}$, we have $\mathcal{Z}(\sigma(t))^\dagger = \{0\}$. Hence, when $y(t) \neq 0$, the optimization problem in (20) has a linear, hence convex, cost whose value is not constant over $\mathcal{Z}(\sigma(t))$. Therefore, using Theorem 3.1 on p.137 of [3], we have $u(t) \in \partial \mathcal{Z}(\sigma(t))$, which implies that one or both of the following hold:

$$u(t) \in \partial \mathcal{V}_2(\sigma(t)) \tag{21}$$

$$u(t) \in \partial \mathcal{U}_1 \tag{22}$$

If $u(t) \notin \partial \mathcal{U}_1$, we have $u(t) \in \partial \mathcal{V}_2(\sigma(t)) \cap \mathrm{int}\mathcal{U}_1$ for some $\sigma(t) \geq 1$, that is, $u(t) \in \mathcal{U}_1$ but $u(t) \notin \mathcal{U}_2$. Hence $u(t) \in \mathcal{U}_1 \setminus \mathcal{U}_2$, i.e., $u(t) \in \mathcal{U}$. Since $x(t) \in \mathcal{X}$ for $t \in [t_0, t_f]$, (x, u) satisfy the dynamics in (5), and $(t_0, t_f, x(t_0), x(t_f)) \in \mathcal{E}$, we conclude $(t_0, t_f, x, u) \in \mathcal{F}_O$. ∎

When the condition (10) does not hold, the Maximum Principle used in the proof of Theorem 2 does not necessarily apply (as discussed in Section 36 in Chapter 6 of [23]). This case requires a different version of the Maximum Principle that is given by Theorem 25 in [23].

In Section V.7 of [2] the concept of a *normal linear system with respect to the set of feasible controls* $\mathcal{V}$ is introduced. It is shown therein that, for a normal linear system, the optimal controls are the extremal points of $\mathcal{V}$ if $\mathcal{V}$ is compact and convex. Theorem 2 gives a weaker condition showing that the optimal control is in a larger subset of the *boundary* of $\mathcal{V}$ than just its extreme points.

4 Convexification of the OCP

This section introduces convex relaxations of several cases of the general OCP given by (1) where the control constraints satisfy the relationships given by (3). For each case, we establish that an optimal solution of the RCP problem is also optimal for the OCP.

Corollary 1 *Suppose the OCP given by (1) with $k = 0$ satisfies Condition 1. If $(t_0, t_f, x, \xi, u, \sigma)$ satisfies Condition 2 and the condition (10), then $(t_0, t_f, x, u) \in \mathcal{F}_O^*$.*

Proof: If $(t_0, t_f, x, \xi, u, \sigma) \in \mathcal{F}_R^*$ and it satisfies the condition (10), then $(t_0, t_f, x, u) \in \mathcal{F}_O$, which follows from Theorem 2. Since $k = 0$, the RCP and OCP have identical cost functions. These imply $\mathbf{J}_O^* \leq \mathbf{J}_R^*$. Since a feasible solution in $\mathcal{F}_O$ defines a feasible solution in $\mathcal{F}_R$ and $k = 0$, $\mathbf{J}_R^* \leq \mathbf{J}_O^*$. Hence $\mathbf{J}_R^* = \mathbf{J}_O^*$, and $(t_0, t_f, x, u) \in \mathcal{F}_O^*$. ∎

The next class of problems has an integral cost on the controls and it is applicable to many minimum fuel planetary soft landing applications [1,15,20], where

$$\mathcal{U} = \{u \in \mathbb{R}^m :\ 1 \leq g_0(u) \leq \rho\}, \tag{23}$$

that is $g_1 = g_0$, $q = 1$, and

$$\mathcal{U}_1 = \{u \in \mathbb{R}^m :\ g_0(u) \leq \rho\}. \tag{24}$$

Then the sets given by (6) and (7) can be shown to be

$$\begin{aligned} \mathcal{V} &= \left\{(u, \sigma) \in \mathbb{R}^{m+1} : \sigma \geq 1,\ g_0(u) \leq \min(\rho, \sigma)\right\} \\ \mathcal{V}_2(\sigma) &= \{u \in \mathbb{R}^m :\ g_0(u) \leq \sigma\}. \end{aligned} \tag{25}$$

Theorem 3 *Suppose that $\mathcal{U}$ satisfies (23) and $k > 0$ for the OCP given by (1). If $(t_0, t_f, x, \xi, u, \sigma) \in \mathcal{F}_R^*$ s.t. $u(t) \in \mathcal{U}$ a.e. $[t_0, t_f]$, then $(t_0, t_f, x, u) \in \mathcal{F}_O^*$.*

Proof: Having $u(t) \in \mathcal{U}$ a.e. $[t_0, t_f]$ implies that $(t_0, t_f, x, u) \in \mathcal{F}_O$ (via Theorem 1) and $1 \leq g_0(u(t)) \leq \sigma(t)$ a.e. $[t_0, t_f]$. Suppose that there exists a nonzero measure set $\mathcal{I} \subset [t_0, t_f]$ such that $g_0(u(t)) < \sigma(t)\ \ \forall t \in \mathcal{I}$. Then $(t_0, t_f, x, \xi, u, \tilde{\sigma}) \in \mathcal{F}_R$, where

$$\tilde{\sigma}(t) = \begin{cases} \sigma(t) & \text{for } t \in \tilde{\mathcal{I}} := [t_0, t_f] \setminus \mathcal{I} \\ g_0(u(t)) & \text{for } t \in \mathcal{I} \end{cases}.$$

Since $\int_{t_0}^{t_f} \tilde{\sigma}(t) dt = \int_{\tilde{\mathcal{I}}} \sigma(t) dt + \int_{\mathcal{I}} g_0(u(t)) dt < \int_{t_0}^{t_f} \sigma(t) dt$, the cost of this new feasible solution is less than $\mathbf{J}_R^*$, which is not possible. Hence $\mathcal{I}$ must have a measure zero, that is, $g_0(u(t)) = \sigma(t)$ a.e. $[t_0, t_f]$ and $\xi(t_f) = \int_{t_0}^{t_f} \sigma(t) dt = \int_{t_0}^{t_f} g_0(u(t)) dt$. This and

$(t_0, t_f, x, u) \in \mathcal{F}_O$ imply that $\mathbf{J}_R^* \geq \mathbf{J}_O^*$. For every solution in $\mathcal{F}_O^*$ with some u, there is a solution in $\mathcal{F}_R$ with the same controls and $\sigma(t) = g_0(u(t))$. Since the corresponding cost of the RCP is the same as $\mathbf{J}_O^*$, $\mathbf{J}_R^* \leq \mathbf{J}_O^*$. Consequently, $\mathbf{J}_R^* = \mathbf{J}_O^*$, and $(t_0, t_f, x, u) \in \mathcal{F}_O^*$. ■

The following corollary gives another condition for an optimal control of the RCP to also define an optimal solution for the OCP.

Corollary 2 *Suppose that $\mathcal{U}$ satisfies (23), $k > 0$, and Condition 1 is satisfied for the OCP given by (1). If $(t_0, t_f, x, \xi, u, \sigma)$ satisfies Condition 2 and the condition (10), then $(t_0, t_f, x, u) \in \mathcal{F}_O^*$.*

Proof: Theorem 2 implies that $(t_0, t_f, x, u) \in \mathcal{F}_O$, and hence $u(t) \in \mathcal{U}$ a.e.$[t_0, t_f]$. Then the proof follows from Theorem 3. ■

Given a feasible state trajectory x on some interval $[t_0, t_f]$ for the OCP or the RCP, let

$$\begin{aligned} &\mathcal{T}_i := \{t : t \in (t_0, t_f) \text{ and } x(t) \in \text{int}\mathcal{X}\} \text{ and} \\ &\mathcal{T}_b := [t_0, t_f] \setminus \mathcal{T}_i, \end{aligned} \tag{26}$$

and the junction times (Section 36 of [23]), $\mathcal{T}_j \subset [t_0, t_f]$,

$$\begin{aligned} \mathcal{T}_j := \{&t \in [t_0, t_f] : \exists s > 0,\ x(\tau) \in \text{int}\mathcal{X} \text{ for either} \\ &\tau \in (t, t+s) \text{ or } \tau \in (t-s, t)\} \end{aligned} \tag{27}$$

A point t of a set $\mathcal{T}$ is called an isolated point, if there exists a neighborhood of t not containing other points of $\mathcal{T}$. A set of isolated points is called a *discrete set* and any discrete subset of an Euclidean space is countable.

Corollary 3 *Suppose that $\mathcal{U}$ satisfies (23), $k > 0$, Condition 1 is satisfied, A, B, and Ew are constant in time for the OCP given by (1). If $(t_0, t_f, x, \xi, u, \sigma)$ satisfies Condition 2 and $\mathcal{T}_j$ in (27) is a discrete set, then $u(t) \in \mathcal{U}$ a.e. $t \in \mathcal{T}_i$. Moreover if $\mathcal{T}_b$ in (26) is a discrete set then $(t_0, t_f, x, u) \in \mathcal{F}_O^*$.*

Proof: Since $\mathcal{T}_j$ is a discrete set, for any consecutive junction times $t_1 \leq t_2$, $x(t) \in \text{int}\mathcal{X}$ for $t \in (t_1, t_2)$ and $x(t_1) \in \partial\mathcal{X}$, $x(t_2) \in \partial\mathcal{X}$. Now pick a large enough integer $a > 0$ such that $t_1 + 1/a < t_2 - 1/a$. Let $t_s := t_1 + 1/a$, $t_e := t_2 - 1/a$, $x_s := x(t_s)$, $x_e := x(t_e)$, and $\xi_s := \xi(t_s)$. Consider a subproblem of the OCP and RCP, where we construct trajectories from a specified x_s at t_s to a specified final state x_e by minimizing the same integral cost defined by g_0. The free variables in the terminal conditions are t_s and $\xi(t_s)$ (which determines the cost). The corresponding portion of the optimal solution of the RCP over the time interval $[t_s, t_e]$ must also be optimal for this subproblem. Otherwise there is another solution of the RCP, $(t_s, t_e^*, x^*, \xi^*, u^*, \sigma^*)$, from x_s to x_e with less cost. Then the modified solution $(t_0, \hat{t}_f, \hat{x}, \hat{\xi}, \hat{u}, \hat{\sigma})$ such that $\hat{t}_f = t_e^* + (t_f - t_e)$ and $(\hat{x}(t), \hat{\xi}(t), \hat{u}(t), \hat{\sigma}(t)) =$

$$\begin{cases} (x(t), \xi(t), u(t), \sigma(t)) & \text{for } t \in [t_0, \hat{t}_f] \setminus [t_s, t_e^*] \\ (x^*(t), \xi^*(t), u^*(t), \sigma^*(t)) & \text{for } \quad\quad t \in [t_s, t_e^*] \end{cases}$$

is also feasible for the main RCP with less cost than $\mathbf{J}_R^*$, which is a contradiction. Since t_e is free and x is fully specified at t_s and t_e, the corresponding $L_t = [\,0\ \ 1\,]^T$ and $L_x = 0$ for the set $\mathcal{E}$ of the subproblem. The subproblem has $k > 0$ and Condition 2 is satisfied (see Remark 1). Also since $x(t) \in \text{int}\mathcal{X}$ for $t \in [t_s, t_e]$ and Condition 1 holds, Corollary 2 implies that $u(t) \in \mathcal{U}$ for $t \in [t_s, t_e]$. Therefore, since $a > 0$ can be arbitrarily large, $u(t) \in \mathcal{U}$ for $t \in (t_1, t_2)$. We have $\mathcal{T}_j = \{t_i,\ i = 1, \ldots\}$, $t_{i+1} > t_i\ \forall i$. Hence $\text{int}\mathcal{T}_i = \cup_i (t_i, t_{i+1})$ and the solution of the RCP has $u(t) \in \mathcal{U}$ a.e. $t \in \mathcal{T}_i$. Next let $\mathcal{T}_b$ be a discrete set with zero measure. Then, $\mathcal{T}_b = \mathcal{T}_j$, $[t_0, t_f] = \text{cl}\mathcal{T}_\text{i}$, and $u(t) \in \mathcal{U}$ a.e. $[t_0, t_f]$. Hence $(t_0, t_f, x, u) \in \mathcal{F}_O^*$, which follows from Theorem 3. ■

Next we extend Corollary 1, by using Corollary 3, to establish lossless convexification when there is no integral cost on the control and a portion of the optimal trajectory of the RCP intersects $\partial\mathcal{X}$. We use a *two-step prioritized optimization* approach:

(1) Consider the OCP (1), satisfying Condition 1, where $k = 0$, $\mathcal{U}$ is given by (23), $A(t) = A$, $B(t) = B$, and $E(t)w(t) = \hat{w}$. We call this problem the *Step-1 OCP* and its convex relaxation, given by (5), the *Step-1 RCP*, which is solved to obtain $(t_0^+, t_f^+, x^+, \xi^+, u^+, \sigma^+)$.

(2) Consider a modified version of the RCP solved in Step 1, where the cost function is $\xi(t_f)$ $(k = 1)$, and the boundary conditions are $(t_0, t_f, x(t_0), x(t_f)) = (t_0^+, t_f^+, x^+(t_0^+), x^+(t_f^+))$. We call this problem the *Step-2 RCP*, which is solved to obtain $(t_0^*, t_f^*, x^*, \xi^*, u^*, \sigma^*)$.

Corollary 4 *Suppose that $\mathcal{U}$ satisfies (23), $k = 0$, Condition 1 is satisfied, and A, B, Ew are constant in time for the OCP given by (1). Suppose that the two-step prioritized optimization approach above is applied. If the Step-1 RCP is feasible, then Step-2 RCP is feasible. If $(t_0^*, t_f^*, x^*, \xi^*, u^*, \sigma^*)$ satisfies Condition 2 and $\mathcal{T}_j$ in (27) is a discrete set, then $u^*(t) \in \mathcal{U}$ a.e. $t \in \mathcal{T}_i$. If in addition $\mathcal{T}_b$ is a discrete set, then $(t_0^*, t_f^*, x^*, u^*) \in \mathcal{F}_O^*$.*

Proof: The only constraint added in going from Step-1 to Step 2 is $(t_0, t_f, x(t_0), x(t_f)) = (t_0^+, t_f^+, x^+(t_0^+), x^+(t_f^+))$. Since the Step-1 RCP's solution also satisfies this constraint, it is feasible for Step-2, proving our first claim. Denote the optimal costs of the Step-1 OCP and RCP as h_0^* and h_0^+. Since the Step-1 RCP is a relaxation of the Step-1 OCP, $h_0^+ \leq h_0^*$. Corollary 3 states that, if $\mathcal{T}_j$ is a discrete set then $u^*(t) \in \mathcal{U}$ a.e. $t \in \mathcal{T}_i$. If $\mathcal{T}_b$ is a discrete set, then (t_0^*, t_f^*, x^*, u^*) is a feasible solution to the Step-1 OCP with the additional constraint $(t_0, t_f, x(t_0), x(t_f)) = (t_0^+, t_f^+, x^+(t_0^+), x^+(t_f^+))$, hence $\bar{h}_0 := h_0(t_0^*, t_f^*, x(t_0)^*, x(t_f)^*) \geq h_0^*$. Since $\bar{h}_0 = h_0^+ \leq h_0^*$, $\bar{h}_0 = h_0^+ = h_0^*$, that is, $(t_0^*, t_f^*, x^*, u^*) \in \mathcal{F}_O^*$. ∎

5 Numerical Examples

An interesting example of the class of problems considered in this paper is the planetary *soft landing* problem [20,10,15,1], where an autonomous vehicle lands at a prescribed location on a planet with zero relative velocity. A simplified version of this problem is described in the form given by (1) with the following parameters: $h_0(t_0, t_f, x(t_0), x(t_f)) = (x(t_0) - z_0)^T Q_0 (x(t_0) - z_0)$, $g_0(u) = \|u\|$, $\mathcal{E}$ is as given by (2), $\mathcal{X} = \{x \in I\!R^4 : \gamma\, |\mathrm{e}_1^T x| \leq \mathrm{e}_2^T x\}$, $\mathcal{U} = \{u \in I\!R^2 : 1 \leq \|u\| \leq \rho\}$, $w = -g\, \mathrm{e}_2$

$$A = \begin{bmatrix} \mathbf{0} & I \\ -\theta^2 I & \theta S \end{bmatrix}, \; B = E = \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix}, \; S = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

with $x = (p_x, p_y, v_x, v_y)$ is the state of two position and two velocity coordinates, $g = 1.5$ is the gravitational acceleration, θ is the planet's rotation rate, $\rho = 4$, and z_0 is the prescribed inital state. Note that, since (A, B) is controllable and $\mathcal{U}_2^\dagger = \{\mathbf{0}\}$, Condition 1 is satisfied. $\mathcal{X}$ defines a cone constraint on the position. The convexification due to using the set of controls $\mathcal{V}$ instead of $\mathcal{U}$ is illustrated in Figure 2 given in Section 3. We use YALMIP [17] with SDPT-3 [24] as the numerical optimizer.

The first example is a minimum distance problem where $\gamma = 1/\sqrt{3}$, $z_0 = (-15, 20, -10, 1)$, $z_f = (0, 0.1, 0, 0)$, t_f is specified, that is, $a_x = (z_0, z_f)$, $a_t = (0, t_f)$, $L_x = \mathrm{e}_1$, $L_t = \mathbf{0}$, $Q = \mathrm{e}_1\, \mathrm{e}_1^T$, for two cases $t_f = 5.5$ and $t_f = 30$. The target is reachable for $t_f = 30$ but not for 5.5. Hence $\partial h_0 / \partial(x(t_0)) = 0$ for the optimal trajectory for $t_f = 30$ but not for $t_f = 5.5$, where the cost is 23.82. Since h_0 only depends on $x(t_0)$ and $k = 0$, none of the conditions in (8) are satisfied for $t_f = 30$. So Corollary 1 does not apply and the simulation results in Figure 3 show that the control bounds are violated. For $t_f = 5.5$, the second condition in (8) is satisfied. Hence Corollary 1 applies and solution of the RCP produces the optimal solution for the OCP (see Figure 3). Note that for both cases the state trajectory lies entirely in int$\mathcal{X}$. The second example has the following changes: $\gamma = 1$ and t_f is free, that is, $a_t = \mathbf{0}$ and $L_t = [0 \;\; 1]^T$. We first solve the minimum distance problem that has an optimal cost of $\rho_* = 4.712$ with $t_f = 17$. Though $\rho_* > 0$, since the solution is on $\partial\mathcal{X}$ for a period of time and since there is no integral cost, the solution shown in Figure 4 violates the control constraints. Next we apply the two step approach of Corollary 4 with $g_0(u) = \|u\|$. Note that, since the cost does not depend on the boundary conditions

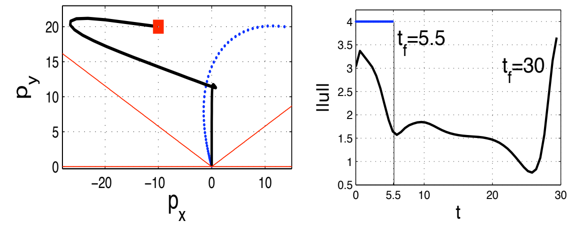

Fig. 3. *Minimum Distance Problem. In $p_x - p_y$ plot: Solid curve indicates solution for $t_f = 30$, and dotted curve for $t_f = 5.5$; The square denotes z_0; The solid straight lines indicates the boundaries of $\mathcal{X}$ on $p_x - p_y$ space.*

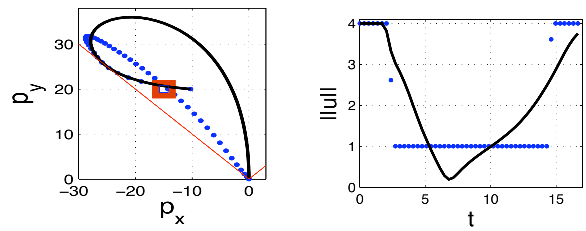

Fig. 4. *Minimum Distance Problem with* prioritization*. In $p_x - p_y$ plot: Solid curve indicates solution of Step-1 (with cost $\rho_* = 4.712$), and dotted curve is for Step-2, where the fuel is minimized; The square denotes z_0;The solid straight lines indicates the boundaries of $\mathcal{X}$ on $p_x - p_y$ space.*

and t_f is free with $\sigma(t_f) \geq 1$, the hypotheses of Corollary 3 are satisfied. Then we find the minimum fuel solution among all minimum distance solutions. The trajectory still intersects with the boundary, but, Corollary 4 states that the convexification holds elsewhere in time. Indeed $u(t) \in \mathcal{U}\ \forall t \in [t_0, t_f]$, hence, via Theorem 3, an optimal solution of the OCP is obtained.

The last example considers a case where the optimal trajectory of the relaxed problem has a significant portion of the trajectory on the boundary. In this example $z_0 = (-10, 20, -10, 1)$, $z_f = \mathbf{0}$, $\gamma = 1/\sqrt{3}$, and t_f is free. The optimal controls for the RCP are in $\mathcal{U}$ for all time, see Figure 5. Since $k > 0$, the resulting trajectory is concluded to be optimal for the OCP by using Theorem 3.

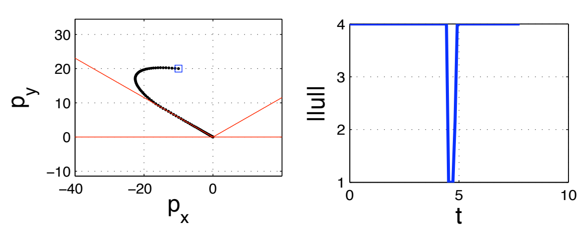

Fig. 5. *Minimum fuel trajectory on the boundary: $u(t) \in \mathcal{U}$ for all $t \in [t_0, t_f]$. Lossless convexification via Theorem 3.*

6 Conclusions

This paper presents a lossless convexification of a finite horizon optimal control problem with convex state and non-convex control constraints. We introduce a convex relaxation of the problem and show that any optimal solution of the relaxed problem, with the state trajectory strictly in the interior of the set of feasible states, is also feasible for the original problem. It is also shown that, these solutions are also optimal for the original problem in some general cases. The results are extended to cases where portions of the optimal state trajectory of the relaxed problem lie on the boundary of the feasible set of states. These results enable us to utilize polynomial time convex programming algorithms to solve the relaxed problem to global optimality to obtain the optimal solutions of the original problem. Hence they can enable real-time optimal path planning algorithms, particularly in the area of planetary landing space missions.

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