Lossless Convexification of a Class of Non-Convex Optimal Control Problems for Linear Systems

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Abstract— **We consider a class of finite time horizon optimal control problems for continuous time linear systems with a convex cost, convex state constraints and non-convex control constraints. We propose a convex relaxation of the non-convex control constraints, and prove that the optimal solution of the relaxed problem is also an optimal solution for the original problem. This lossless convexification approach enables the use of interior point methods of convex optimization to obtain globally optimal solutions of the original non-convex optimal control problem. We demonstrate this solution approach with a number of planetary soft landing problems.**

I. Introduction

This paper considers a class of finite time horizon optimal control problems with a convex cost, convex state and non-convex control constraints. There are a variety of optimal control problems that fall into this class. One interesting example, which is also the motivating example of this paper, is the planetary landing problem[11], [8], also known as the soft landing problem in the optimal control literature[6]. In planetary landing, an autonomous spacecraft lands on the surface of a planet by using thrusters, which can produce a finite magnitude force vector with an upper and nonzero lower bounds on the magnitude. The nonzero lower bound constraint exists due to the fact that the thrusters cannot operate reliably under this bound; if the thrusters are throttled below this value, they may not restart, leading to failure of the mission. This constraint on the magnitude makes the set of feasible controls non-convex. There are also convex state constraints that ensure, among other things, that the spacecraft does not go below the surface of the planet.

The non-convexity of an optimal control problem can be due to: (i) non-convex cost; (ii) nonlinear state dynamics; (iii) non-convex state constraints; and (iv) non-convex control constraints. Since the systems considered in this paper have convex cost, linear system dynamics and convex state constraints, we only have a single source of non-convexity, namely the control constraints. In this paper, we introduce a *convex relaxation* of the control constraints. This convex relaxation proceeds as follows. The non-convex set of feasible control inputs is replaced by a convex set of feasible control inputs by introducing a slack variable. It is first shown that the optimal solution of the relaxed problem also defines a feasible solution of the original one when the optimal state trajectory of the relaxed problem is strictly in the interior of the set of feasible states. It is then shown that under further assumptions, the optimal trajectories of the relaxed problem are also optimal for the original problem. Hence a *lossless convexification* can be achieved. Here, we use lossless convexification to refer to obtaining a convex relaxation of an optimization problem, where an optimal solution of the relaxed problem also defines an optimal solution for the original non-convex problem. These results are also extended to cases where some portions of the optimal trajectories for the relaxed problem lie on the boundary. The convexification allows the relaxed problem to be posed as a finite dimensional convex programming problem, which can be obtained after a discretization. A generic convex programming problem, under mild computability and regularity assumptions, is solvable in polynomial time and hence it is computationally tractable [12], [4].

Given the lossless theoretical convexification results, a class of non-convex optimal control problems, including planetary soft landing problems[11], [8], can be solved by using polynomial time algorithms developed for convex programming [12], [4], [14]. In our previous work we demonstrated a special case of our convexification result for Mars landing[1]. The present paper extends this work to a more general class of control problems. Finally, our convexification can enable the real-time application of model predictive control techniques to obtain a robust feedback control action for these problems [7], [10].

Notation

The following is a partial list of notation used in this paper: $Q = Q^T > (\geq) 0$ implies that Q is a symmetric positive (semi-)definite matrix; $\mathbb{R}$ is the set of real numbers; A condition is said to hold almost everywhere in the interval $[a, b]$, a.e. $[a, b]$, if the set of points in $[a, b]$ where this condition fails to hold is contained in a set of measure zero; $\mathbb{R}^n$ is the n dimensional real vector space; $\emptyset$ denotes the empty set; $\|v\|$ is the 2-norm of the vector v; $\|v\|_p$ is the p-norm of the vector v; $\mathbf{0}$ is matrix of zeros with appropriate dimension; I is the identity matrix; $\mathbf{1}$ is the matrix of ones with appropriate dimensions; $\mathbf{e}_i$ is a vector of appropriate dimension with its ith entry $+1$ and its other entries zeros; $(v_1, v_2, ..., v_m)$ represents a vector obtained by augmenting vectors $v_1, \ldots, v_m$ such that:

$$(v_1, v_2, ..., v_m) \equiv \begin{bmatrix} v_1^T & v_2^T & \ldots & v_m^T \end{bmatrix}^T.$$

where v_i have arbitrary dimensions. We use $\partial\mathcal{S}$ to denote the set of boundary points of the set $\mathcal{S}$, and use $\mathbb{R}_+$ to denote

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the extended real numbers set that includes $\pm\infty$. Finally, $U^{\perp}$ is defined for a set $U \subset \mathbb{R}^n$ as follows:

$$U^{\perp} := \left\{ v \in \mathbb{R}^n : \ \exists c \in \mathbb{R} \ \text{ s.t. } \ \mathrm{v^T u = c} \quad \forall \mathrm{u} \in \mathcal{U} \right\}.$$

II. Problem Formulation

This section introduces the following finite time horizon optimal control problem considered in this paper. The assumptions introduced here are used throughout the paper.

Original Optimal Control Problem

$$\min_{\omega_t, \omega_x, u(\cdot)} \ h_0(t_0, t_f, x(t_0), x(t_f)) + k \int_{t_0}^{t_f} g_0(u(t))\, dt$$
subject to
$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + E(t)w(t)$$
$$x(t) \in \mathcal{X} \ \text{ and } \ u(t) \in \mathcal{U} \quad \text{a.e. } [t_0, t_f]$$
$$(t_0, t_f, x(t_0), x(t_f)) \in \mathcal{E} \tag{1}$$

where t is the time, t_0 is the initial time, t_f is the finite final time, $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, w is a known exogenous input, $h_0 : \mathbb{R}^{2n} \to \mathbb{R}$ is a convex function describing the cost on the end states, $g_0 : \mathbb{R}^m \to \mathbb{R}$ is a convex function describing the integral cost on the control input, $k \geq 0$ is a scalar, $A : \mathbb{R}_+ \to \mathbb{R}^{\mathrm{n \times n}}$ and $B : \mathbb{R}_+ \to \mathbb{R}^{\mathrm{n \times m}}$ are piecewise analytic functions of time, $\mathcal{X} \subseteq \mathbb{R}^n$ is the set of feasible states, $\mathcal{U} \in \mathbb{R}^m$ is the set of feasible control inputs, and $\mathcal{E} \subset \mathbb{R}^{2n+2}$ is the set of feasible *boundary conditions*. It is assumed that $\mathcal{X}$ is a convex set and $\mathcal{U}$ is a non-convex set such that:

$$\mathcal{U} = \mathcal{U}_1 \setminus \mathcal{U}_2, \qquad \mathcal{U}_2 = \bigcap_{i=1}^{q} \mathcal{U}_{2,i} \ \subset \ \mathcal{U}_1, \tag{2}$$

where $\mathcal{U}_1$ is a compact convex set and $\mathcal{U}_2$ is an open convex set with:

$$\mathcal{U}_{2,i} = \{ u \in \mathbb{R}^m : \ g_i(u) < 1 \}, \quad i = 1, ..., q, \tag{3}$$

with $g_i, \ i = 1, ..., q$, are convex functions that are bounded on $\mathcal{U}_1$, that is, there exists some $\bar{g} \in \mathbb{R}$ such that $g_i(u) \leq \bar{g}, \ \forall u \in \mathcal{U}_1, \ i = 1, ..., q$. Note that $\mathcal{U}_2 \cap \partial \mathcal{U}_1$ is empty. This follows from the fact that $\mathcal{U}_2 \subset \mathcal{U}_1$ and $\mathcal{U}_2 \cap \partial \mathcal{U}_2$ is empty. It is assumed that the set of boundary conditions $\mathcal{E}$ defines a smooth manifold and any tangent hyperplane to this manifold contains the hyperplane $\mathcal{E}_L$ defined as follows:

$$\mathcal{E}_L := \left\{ \begin{bmatrix} t_0 \\ t_f \\ x_0 \\ x_f \end{bmatrix} \in \mathbb{R}^{2n+2} : \begin{bmatrix} t_0 \\ t_f \\ x_0 \\ x_f \end{bmatrix} = \begin{bmatrix} a_t \\ a_x \end{bmatrix} + \begin{bmatrix} L_t & \mathbf{0} \\ \mathbf{0} & L_x \end{bmatrix} \begin{bmatrix} \omega_t \\ \omega_x \end{bmatrix}, \ (\omega_t, \omega_x) \in \mathbb{R}^{q_e} \right\}, \tag{4}$$

where ω_t and ω_x are free variables describing the degrees of freedom in the initial and final states and times, a_x and a_t are prescribed boundary conditions and L_x and L_t are matrices satisfying $L_x^T L_x = I$ and $L_t^T L_t = I$. Note that if all boundary conditions are prescribed, $q_e = 0$.

III. Main Technical Result

This section presents the main theoretical result, given by Theorem 2, that leads to the convexification results in this paper. The result is based on the following relaxed version of the optimal control problem (1):

Relaxed Optimal Control Problem

$$\min_{\omega_t, \omega_x, \omega_\xi, u(\cdot), \sigma(\cdot)} \ h_0(t_0, t_f, x(t_0), x(t_f)) + k\, \xi(t_f)$$
subject to
$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + E(t)w(t)$$
$$\dot{\xi}(t) = \sigma(t)$$
$$x(t) \in \mathcal{X} \ \text{ and } \ (u(t), \sigma(t)) \in \mathcal{V} \quad \text{a.e. } [t_0, t_f]$$
$$(t_0, t_f, x(t_0), x(t_f), \xi(t_0), \xi(t_f)) \in \tilde{\mathcal{E}} := \mathcal{E} \times \{0\} \times \mathbb{R}_+ \tag{5}$$

where $\sigma \in \mathbb{R}$ is a slack control variable, $\xi \in \mathbb{R}$ is a slack state variable, and:

$$\mathcal{V} = \left\{ (u, \sigma) \in \mathbb{R}^{m+1} : \sigma \geq 1 \text{ and } u \in \mathcal{U}_1 \cap \mathcal{V}_2(\sigma) \right\} \tag{6}$$

$$\text{with } \ \mathcal{V}_2(\sigma) = \bigcap_{i=i_o}^{q} \mathcal{V}_{2,i}(\sigma) \quad \text{where}$$
$$\mathcal{V}_{2,i}(s) := \{ u \in \mathbb{R}^m : \ g_i(u) \leq s \}, \tag{7}$$
$$i_o = \begin{cases} 0 & \text{for} \quad k > 0 \\ 1 & \text{for} \quad k = 0 \end{cases}.$$

We illustrate the set of feasible control inputs of the relaxed problem, $\mathcal{V}$, above versus the set $\mathcal{U}$ of the original problem (1) on an example in Figure 1. This illustration applies to the planetary soft landing example described in detail in Section V. Here, $\mathcal{V}$ belongs to a higher dimensional space and $\mathcal{U} \subset \mathcal{V}$. Furthermore $\mathcal{V}$ also contains control inputs that are not feasible for the original problem, that is, there exists some $(\hat{u}, \hat{\sigma}) \in \mathcal{V}$ such that $\hat{u} \notin \mathcal{U}$. Hence it is not trivial to establish that the optimal solutions of the relaxed problem will also define feasible solutions for the original problem. Note that if the optimal solutions of the relaxed problem have $(u, \sigma) \in \partial \mathcal{V}$ then $u \in \mathcal{U}$.

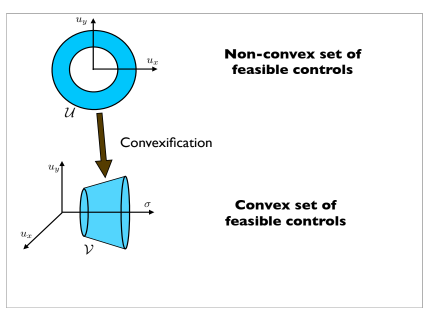

Fig. 1. *Convexification of the Control Magnitude Constraint for Soft Landing. The annulus represents the actual non-convex control constraints in $u_x - u_y$ space, which is lifted to a convex cone in $u_x - u_y - \sigma$ space for the relaxed control problem (5).*

In the following results we denote a feasible solution of the original problem by $(t_0, t_f, x(\cdot), u(\cdot))$, and a feasible solution of the relaxed problem by $(t_0, t_f, x(\cdot), \xi(\cdot), u(\cdot), \sigma(\cdot))$, where $x(\cdot)$ represents the state trajectory for $t \in [t_0, t_f]$ and so on. Unless otherwise stated, a *solution* of the either optimal control problem above refers to an *optimal solution* of the problem. When we refer to a *feasible solution* of either problem, we imply that the solution satisfies the differential equations, and the state and the control constraints of the problem almost everywhere on $[t_0, t_f]$, but it is not necessarily optimal. We refer to any of these problems as *feasible* when the problem has a feasible solution.

Our first result states some apparent facts about the relationship between both optimal control problems.

Theorem 1: Consider the original optimal control problem given by (1) and its convex relaxation given by (5). Then following hold:

(i) If $(t_0, t_f, x(\cdot), u(\cdot))$ is a feasible solution of the original problem, then there exists $(\xi(\cdot), \sigma(\cdot))$ such that $(t_0, t_f, x(\cdot), \xi(\cdot), u(\cdot), \sigma(\cdot))$ is a feasible solution of the relaxed problem.

(ii) If $(t_0, t_f, x(\cdot), \xi(\cdot), u(\cdot), \sigma(\cdot))$ is a feasible solution of the relaxed problem such that $u(t) \in \mathcal{U}$ a.e. $[t_0, t_f]$, then $(t_0, t_f, x(\cdot), u(\cdot))$ is feasible solution of the original problem.

Proof: Suppose that $(t_0, t_f, x(\cdot), u(\cdot))$ is a feasible solution of the original problem. Let

$$\sigma(t) = \max_{i=1,\dots,q} g_i(u(t)) \quad \text{and} \quad \dot{\xi}(t) = \sigma(t) \text{ with } \xi(t_0) = 0,$$

which implies that $\sigma(t) \geq 1$. Clearly $(u(t), \sigma(t)) \in \mathcal{V}$, as well as $x(t) \in \mathcal{X}$ a.e. $[t_0, t_f]$. Consequently $(t_0, t_f, x(\cdot), \xi(\cdot), u(\cdot), \sigma(\cdot))$ is a feasible solution of the relaxed problem, which proves the first claim.

Next suppose that $(t_0, t_f, x(\cdot), \xi(\cdot), u(\cdot), \sigma(\cdot))$ is a feasible solution of the relaxed problem such that $u(t) \in \mathcal{U}$ a.e. $[t_0, t_f]$. Since $x(t) \in \mathcal{X}$ a.e. $[t_0, t_f]$, $(t_0, t_f, x(\cdot), u(\cdot))$ is a feasible solution of the original problem, which completes the proof of the second claim. ■

The following conditions are instrumental in proving our next result.

Condition 1: The pair $\{A(\cdot), B(\cdot)\}$ is *controllable* and the set of feasible controls $\mathcal{U}$ satisfies that $\mathcal{U}_2^{\perp} = \{\mathbf{0}\}$.

Condition 2: $(t_0, t_f, x(\cdot), \xi(\cdot), u(\cdot), \sigma(\cdot))$ is an optimal solution of the relaxed optimal control problem (5) and it satisfies

$$\left(-k\sigma(t_0) - \frac{\partial h_0}{\partial t_0},\ k\sigma(t_f) - \frac{\partial h_0}{\partial t_f},\ \frac{\partial h_0}{\partial x(t_0)},\ \frac{\partial h_0}{\partial x(t_f)}\right) \quad \text{is not orthogonal to } \mathcal{E}_L, \tag{8}$$

where $\mathcal{E}_L$ is given by (4).

Remark 1: A discussion of Condition 2: Lemma 2 in the Appendix implies that a vector $(\phi_0, \phi_f, \psi_0, \psi_f)$ is not orthogonal to $\mathcal{E}_L$ if and only if $(\phi_0, \phi_f, \psi_0, \psi_f) \notin \mathcal{E}_L^{\perp}$, which is also equivalent to having one of the following conditions hold

$$L_t^T \begin{bmatrix} \phi_0 \\ \phi_f \end{bmatrix} \neq 0 \quad \text{or} \quad L_x^T \begin{bmatrix} \psi_0 \\ \psi_f \end{bmatrix} \neq 0. \tag{9}$$

In some cases it is straightforward to see that any feasible solution of the relaxed or the original problem satisfies this condition. For example suppose that t_0 is fixed and t_f is a free variable, which implies that $L_t = [0 \ \ 1]^T$. If h_0 is not a function of t_f, that is, if $\partial h_0 / \partial t_f = 0$, and $k > 0$, which implies that $k\sigma(t_f) > 0$. Then, by letting $\phi_0 = -k\sigma(t_0) - \partial h_0/\partial t_0$ and $\phi_f = k\sigma(t_f) - \partial h_0/\partial t_f$, we can show that the first condition in (9) is satisfied. This example typically arises in the minimum fuel optimal control problems.

Remark 2: A discussion of $\mathcal{U}_2^{\perp} = \{\mathbf{0}\}$: This condition can be quite straightforward to verify in many cases. For example, suppose that there is an interval contained in $\mathcal{U}_2$ along any given direction, that is, for any $\hat{v} \in \mathbb{R}^m$, $\|\hat{v}\| = 1$, there exist some $a < b$ such that $V(\hat{v}) := \{u \in \mathbb{R}^m : u = \lambda \hat{v},\ a \leq \lambda \leq b\} \subset \mathcal{U}_2$. Note that any set $\mathcal{U}_2$ with finite volume in $\mathbb{R}^m$ satisfies this property. In this case $\mathcal{U}_2^{\perp} = \{\mathbf{0}\}$.

Our next theorem presents a fundamental result that establishes conditions under which $u(t) \in \mathcal{U}$ a.e. $[t_0, t_f]$, when $u(\cdot)$ is obtained from the solution of the relaxed problem.

Theorem 2: Consider the optimal control problem given by (1) and its convex relaxation given by (5). Suppose that Condition 1 holds. If an optimal solution $(t_0, t_f, x(\cdot), \xi(\cdot), u(\cdot), \sigma(\cdot))$ of the relaxed problem satisfies Condition 2 and

$$x(t) \in \mathrm{int}\mathcal{X}, \quad \forall\, \mathrm{t} \in [\mathrm{t_0}, \mathrm{t_f}], \tag{10}$$

then $(t_0, t_f, x(\cdot), u(\cdot))$ is a feasible solution of the original problem.

Proof: Let $(t_0, t_f, x(\cdot), \xi(\cdot), u(\cdot), \sigma(\cdot))$ be an optimal solution of the relaxed control problem (5). Since the condition (10) holds, by using the Maximum principle (see Section V.3 of [2] or Chapter 1 of [13]), there exist a constant $\alpha \leq 0$ and absolutely continuous vector functions $\lambda(\cdot)$ and $\eta(\cdot)$, which will be referred as the *co-state vectors*, on $[t_0, t_f]$ such that the following conditions hold:

(i) *Nonzero Co-states:*

$$\mu(t) := \begin{bmatrix} \alpha \\ \lambda(t) \\ \eta(t) \end{bmatrix} \neq 0, \qquad \forall\ t \in [t_0, t_f]. \tag{11}$$

(ii) *Co-state Dynamics*:

$$\begin{aligned} \dot{\lambda}(t) &= -A(t)^T \lambda(t) \\ \dot{\eta}(t) &= 0 \end{aligned} \qquad \text{a.e. } t \in [t_0, t_f] \tag{12}$$

(iii) *Pointwise Maximum Principle*: For a.e. $t \in [t_0, t_f]$,

$$H(t, x(t), \xi(t), u(t), \sigma(t), \mu(t)) = M(t, x(t), \xi(t), \mu(t)) \tag{13}$$

where H is the *Hamiltonian* defined by

$$H(t, x, \xi, u, \sigma, \mu) := \eta\sigma + \lambda^T \left[A(t)x + B(t)u + E(t)w(t)\right], \tag{14}$$

and

$$M(t, x, \xi, \mu) := \max_{(u,\sigma) \in \mathcal{V}} H(t, x, \xi, u, \sigma, \mu). \tag{15}$$

(iv) *Transversality Condition*:

$$L_x^T \begin{bmatrix} -\lambda(t_0) - \alpha \dfrac{\partial h_0}{\partial x(t_0)} \\ \lambda(t_f) - \alpha \dfrac{\partial h_0}{\partial x(t_f)} \end{bmatrix} = \mathbf{0},$$

$$L_t^T \begin{bmatrix} H(\phi(t_0)) - \alpha \dfrac{\partial h_0}{\partial t_0} \\ -H(\phi(t_f)) - \alpha \dfrac{\partial h_0}{\partial t_f} \end{bmatrix} = \mathbf{0}, \;\; \eta(t_f) = -\alpha k \;\; \text{where} \tag{16}$$

$$\phi(t) := (t, x(t), \xi(t), u(t), \sigma(t), \alpha, \lambda(t), \eta(t)) . \tag{17}$$

The necessary conditions of optimality from (i)-(iii) are directly obtained by using the statement of the pointwise maximum principle, where we use the fact that $\mathcal{U}$ is a constant set, i.e., it does not depend on time or state. However the transversality condition given in (iv) requires further explanation. The transversality condition, for an optimal solution of the relaxed problem, states that (see Sec. V.3 of [2]) the vector $\psi(\phi(t_0), \phi(t_f))$ defined as follows

$$\Big(H(\phi(t_0)) - \alpha \frac{\partial h_0}{\partial t_0}, \; -H(\phi(t_f)) - \alpha \frac{\partial h_0}{\partial t_f}, -\lambda(t_0) - \alpha \frac{\partial h_0}{\partial x(t_0)}, \\ \lambda(t_f) - \alpha \frac{\partial h_0}{\partial x(t_f)}, -\eta(t_0), \; -\eta(t_f) - \alpha k \Big)$$

must be orthogonal to the manifold defined by the set of feasible boundary conditions $\tilde{\mathcal{E}}$ at $(\phi(t_0), \phi(t_f))$. Given any point $(t_0, t_f, x(t_0), x(t_f), \xi(t_0), \xi(t_f)) \in \tilde{\mathcal{E}}$, by using the condition (4), the corresponding tangent plane to the manifold defined by $\tilde{\mathcal{E}}$ at that point contains the following hyperplane

$$\tilde{\mathcal{E}}_L = \mathcal{E}_L \times \left\{ z \in \mathbb{R}^2 : \; z = \kappa \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \;\; \kappa \in \mathbb{R} \right\}.$$

Consequently a vector is orthogonal to the tangent hyperplane of the manifold defined by the set $\tilde{\mathcal{E}}$ only if it is also orthogonal to $\tilde{\mathcal{E}}_L$. A vector $v \in \tilde{\mathcal{E}}_L$ if and only if it can be expressed as follows

$$v = \underbrace{\begin{bmatrix} a_t \\ a_x \\ 0 \\ 0 \end{bmatrix}}_{\tilde{a}} + \underbrace{\begin{bmatrix} L_t & \mathbf{0} & 0 \\ \mathbf{0} & L_x & 0 \\ \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & 1 \end{bmatrix}}_{T} \begin{bmatrix} \omega_t \\ \omega_x \\ \omega_\xi \end{bmatrix}$$

for some ω_t, ω_x, ω_ξ, that is,

$$\tilde{\mathcal{E}}_L = \{ v \in \mathbb{R}^{2n+4} : \; v = \tilde{a} + T\omega, \;\; \omega \in \mathbb{R}^{q_e+1} \}.$$

Hence, by using Lemma 2 in the Appendix, $\psi(\phi(t_0), \phi(t_f))$ defined above is orthogonal to $\tilde{\mathcal{E}}_L$ if and only if $T^T \psi(\phi(t_0), \phi(t_f)) = 0$, which is equivalent to the equation (16). Now we claim that

$$y(t) := B(t)^T \lambda(t) \neq 0 \qquad \text{a.e. } t \in [t_0, t_f]. \tag{18}$$

Since A is a piecewise analytic matrix valued function of time, λ is piecewise analytic over $[t_0, t_f]$ (see Theorem 3 on p.213 of [5]). Consequently, since the product of analytic functions is also analytic, y is also piecewise analytic over the time interval. Hence y has a finite number of intervals, $[t_1, t_2] \subset [t_0, t_f]$, in which either $y(t) = 0 \; \forall y \in [t_1, t_2]$, or the set of $t \in [t_1, t_2]$ for which $y(t) = 0$ is countable (see Proposition 4.1 on p.41 of [5]). We now show that none of these intervals has $y(t) = 0 \; \forall y \in [t_1, t_2]$.

Let us assume that there exists an interval $[t_1, t_2] \subset [t_0, t_f]$ such that $y(t) = 0 \; \forall t \in [t_1, t_2]$. First note that the transversality condition (16) implies that $\eta(t_f) = -\alpha k$. Since η is absolutely continuous with $\dot{\eta} = 0$, by using (16), this implies that $\eta(t) = -\alpha k \; \forall t \in [t_0, t_f]$. Since the pair $\{A(\cdot), B(\cdot)\}$ is controllable, it follows from Lemma 1 in the Appendix and $\eta = -\alpha k$ that $\lambda(t) = 0 \; \forall t \in [t_0, t_f]$. Hence the transversality condition (16) implies that:

$$\alpha L_x^T \begin{bmatrix} \dfrac{\partial h_0}{\partial x(t_0)} \\ \dfrac{\partial h_0}{\partial x(t_f)} \end{bmatrix} = 0 \;\; \text{and} \;\; \alpha L_t^T \begin{bmatrix} -k\sigma(t_0) - \dfrac{\partial h_0}{\partial t_0} \\ k\sigma(t_f) - \dfrac{\partial h_0}{\partial t_f} \end{bmatrix} = 0. \tag{19}$$

Note that, by using Lemma 2 in the Appendix, $\mathcal{E}_L$ is a hyperplane with:

$$\mathcal{E}_L^\perp = \left\{ (s, z) : \; L_t^T s = 0, \;\; L_x^T z = 0 \right\}. \tag{20}$$

This, combined with the equalities in (19), implies that $\alpha = 0$ when the condition given by (8) holds. As a result we have: $(\alpha, \, \lambda(t), \, \eta(t)) = 0$ a.e. $t \in [t_0, t_f]$. Since the solution must satisfy the necessary condition (11), this is a contradiction, which resulted from assuming that there exists an interval $[t_1, t_2]$ such that $y(t) = 0 \; \forall t \in [t_1, t_2]$. Hence no such interval exists, which implies that for every interval the set of $t \in [t_1, t_2]$ for which $y(t) = 0$ is countable. This, combined with the fact that there is a finite number of such intervals on $t \in [t_0, t_f]$, implies that the set $Z = \{t : \; y(t) = 0, \;\; t \in [t_0, t_f]\}$ is a countable subset of $[t_0, t_f]$ (see Proposition 4.1 on p.41 of [5]). Since a countable set has a measure zero, this shows that $y(t) \neq 0$ a.e. $t \in [t_0, t_f]$.

Next we will prove that the optimal controls can only be on the boundary of the feasible set. The pointwise maximum principle implies that the optimal controls must satisfy, for a.e. $t \in [t_0, t_f]$,

$$y(t)^T u(t) - \alpha k \sigma(t) = \max_{(z, v) \in \mathcal{V}} y(t)^T z - \alpha k v.$$

Hence, the optimal control pair $\{u(t), \sigma(t)\}$ must satisfy:

$$u(t) = \operatorname*{argmax}_{z \in \mathcal{Z}(\sigma(t))} y(t)^T z \text{ where } \mathcal{Z}(\sigma(t)) := \mathcal{U}_1 \cap \mathcal{V}_2(\sigma(t)). \tag{21}$$

Since $\mathcal{U}_2 \subset \mathcal{Z}(\sigma(t))$ for $\sigma(t) \geq 1$ and $\mathcal{U}_2^\perp = \{0\}$, we have $\mathcal{Z}(\sigma(t))^\perp = \{0\}$. Hence, when $y(t) \neq 0$, the optimization problem (21) has a cost whose value is not constant over $\mathcal{Z}(\sigma(t))$. Since the cost is not constant and it is convex on $\mathcal{Z}(\sigma(t))$, using Theorem 3.1 on p.137 of [3], $u(t) \in \partial \mathcal{Z}(\sigma(t))$. This implies that one or both of the following hold:

$$u(t) \in \partial \mathcal{V}_2(\sigma(t)) \tag{22}$$

$$u(t) \in \partial \mathcal{U}_1 \tag{23}$$

Hence if $u(t) \notin \partial\mathcal{U}_1$, we have $u(t) \in \partial\mathcal{V}_2(\sigma(t)) \cap \text{int}\mathcal{U}_1$ for some $\sigma(t) \geq 1$, which then implies that $u(t) \in \mathcal{U}_1$ but $u(t) \notin \mathcal{U}_2$. Consequently $u(t) \in \mathcal{U}_1 \setminus \mathcal{U}_2$, which implies that $u(t) \in \mathcal{U}$. Since $x(t) \in \mathcal{X}$ for $t \in [t_0, t_f]$, $(t_0, t_f, x(\cdot), u(\cdot))$ is a feasible solution of the original problem. ∎

IV. Convexification of the Optimal Control Problem

This section introduces convex relaxations of several cases of the general optimal control problem given by (1). For each case, we establish that an optimal solution of the relaxed control problem is also optimal for the original problem.

Corollary 1: Consider the original optimal control problem given by (1) with $k = 0$ and its convex relaxation given by (5). Suppose that Condition 1 holds. If $(t_0, t_f, x(\cdot), \xi(\cdot), u(\cdot), \sigma(\cdot))$ is an optimal solution of the relaxed problem satisfying Condition 2 and the condition given by (10), then $(t_0, t_f, x(\cdot), u(\cdot))$ is an optimal solution for the original problem (1).

The next control problem has an integral cost on the control effort and it is applicable to many soft landing applications where fuel use must be minimized [1], [8], [11]. We assume that the set of feasible controls is given by

$$\mathcal{U} = \{u \in \mathbb{R}^m : \ 1 \leq g_0(u) \leq \rho\}, \tag{24}$$

that is $g_1 = g_0$, $q = 1$, and:

$$\mathcal{U}_1 = \{u \in \mathbb{R}^m : \ g_0(u) \leq \rho\}. \tag{25}$$

Theorem 3: Consider the optimal control problem (1) and its convex relaxation given by (5), where $\mathcal{U}$ satisfies (24) and $k > 0$. If $(t_0, t_f, x(\cdot), \xi(\cdot), u(\cdot), \sigma(\cdot))$ is an optimal solution of the relaxed problem such that $u(t) \in \mathcal{U}$ a.e. $[t_0, t_f]$, then $(t_0, t_f, x(\cdot), u(\cdot))$ is an optimal solution of the original problem.

Proof: Since $u(t) \in \mathcal{U}$ a.e. $[t_0, t_f]$, $1 \leq g_0(u(t)) \leq \sigma(t) \leq \rho$ a.e. $[t_0, t_f]$. Suppose that there exists $\mathcal{I} \subset [t_0, t_f]$ such that $\mathcal{I}$ is a set of nonzero measure and $g_0(u(t)) < \sigma(t)$ for $t \in \mathcal{I}$. This implies that $(t_0, t_f, x(\cdot), \xi(\cdot), u(\cdot), \tilde{\sigma}(\cdot))$ is a feasible solution of the relaxed problem, where:

$$\tilde{\sigma}(t) = \begin{cases} \sigma(t) & \text{for } t \in \tilde{\mathcal{I}} := [t_0, t_f] \setminus \mathcal{I} \\ g_0(u(t)) & \text{for } t \in \mathcal{I} \end{cases} .$$

Furthermore since: $\int_{t_0}^{t_f} \tilde{\sigma}(t)dt = \int_{\tilde{\mathcal{I}}} \sigma(t)dt + \int_{\mathcal{I}} g_0(u(t))dt < \int_{t_0}^{t_f} \sigma(t)dt$, the cost of this new solution is less than the optimal solution, which is not possible. Hence $\mathcal{I}$ must have a measure zero, that is, $g_0(u(t)) = \sigma(t)$ a.e. $[t_0, t_f]$. This then implies that the cost function in the relaxed problem is identical to the cost function in the original problem, since: $\xi(t_f) = \int_{t_0}^{t_f} \sigma(t)dt = \int_{t_0}^{t_f} g_0(u(t))dt$. Note that $1 \leq g_0(u(t)) = \sigma(t) \leq \rho$, which implies that $u(t) \in \mathcal{U}$ a.e. $[t_0, t_f]$. Then, following from Theorem 1 $(t_0, t_f, x(\cdot), u(\cdot))$ is feasible for the original problem. Since the cost functions are identical and the optimal control for the relaxed problem is feasible for the original problem, we know that the cost of the optimal solution to the relaxed problem is greater than or equal to the optimal cost of the original problem. Next for every feasible solution of the original problem (1) with the optimal control input $u(\cdot)$, there exists a feasible solution of the relaxed problem (5) with the same control input and $\sigma(t) = g_0(u(t))$. Since the corresponding cost function of the relaxed problem is the same as the cost function of the original problem, this means that the optimal cost of the relaxed problem is equal to, or less than, the optimal cost of the original problem. From the above discussion, the optimal solution to the relaxed problem also gives an optimal solution of the original problem, which completes the proof. ∎

Corollary 2: Consider the optimal control problem (1) and its convex relaxation given by (5), where $\mathcal{U}$ satisfies (24), $k > 0$, and $h_0 = 0$. Suppose that Condition 1 is satisfied. Then, if $(t_0, t_f, x(\cdot), \xi(\cdot), u(\cdot), \sigma(\cdot))$ is an optimal solution of the relaxed problem satisfying Condition 2 and the condition given by (10), then $(t_0, t_f, x(\cdot), u(\cdot))$ is an optimal solution of the original problem.

V. Numerical Examples

An interesting example to the class of problems considered in this paper is the planetary *soft landing* problem [11], [8], [1], where an autonomous vehicle lands at a prescribed location on a planet with zero relative velocity. A simplified version of this problem is described in the form given by (1) with the following parameters: $h_0(t_0, t_f, x(t_0), x(t_f)) = (x(t_0) - z_0)^T Q_0 (x(t_0) - z_0)$, $g_0(u) = \|u\|$, $\mathcal{E}$ is as given by (4), $\mathcal{X} = \{x \in \mathbb{R}^4 : \gamma \, |\mathbf{e}_1^T x| \leq \mathbf{e}_2^T x\}$, $\mathcal{U} = \{u \in \mathbb{R}^2 : \ 1 \leq \|u\| \leq \rho\}$,

$$A = \begin{bmatrix} \mathbf{0} & I \\ -\theta^2 I & \theta S \end{bmatrix}, \ B = E = \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix}, \ w = -g\mathbf{e}_2, \ S = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

with $x = (p_x, p_y, v_x, v_y)$ is the state of two position and two velocity coordinates, $g = 1.5$ is the gravitational acceleration, θ is the planet's rotation rate, $\rho = 4$, and z_0 is thez prescribed inital state.Note that, since (A, B) is controllable and $\mathcal{U}_2^{\perp} = \{\mathbf{0}\}$, Condition 1 is satisfied. $\mathcal{X}$ defines a cone constraint on the position.The convexification due to using the set of controls $\mathcal{V}$ instead of $\mathcal{U}$ is illustrated in Figure 1 given in Section III. We use YALMIP [9] with SDPT-3 [14] as the numerical optimizer.

In the first set of simulations we consider a minimum distance problem where $g = 1.5$, $p = 1$, $\gamma = 1/\sqrt{3}$, $z_0 = (-15, 20, -10, 1)$, t_f is specified, that is, $a_x = (z_0, z_f)$, $a_t = (0, t_f)$, $L_x = \mathbf{e}_1$ $L_t = \mathbf{0}$, $w_t = 0$, $\hat{t} = t_f$, $Q = \mathbf{e}_1\mathbf{e}_1^T$, with $z_f = (0, 0.1, 0, 0)$. Here we compare two cases with different maneuver times, one with $t_f = 5.5$ and the other with $t_f = 30$. The target is reachable for $t_f = 30$, hence the optimal value of the cost is zero, that is, $\partial h_0/\partial(x(t_0)) = 0$ for the optimal trajectory. Since h_0 only depends on $x(t_0)$ and $k = 0$, none of the conditions in (8) are satisfied. The simulation results given in Figure 2 show that the control bounds are violated for $t_f = 30$. For $t_f = 5.5$ the value of the optimal cost is 23.82 and hence $\partial h_0/\partial(x(t_0)) \neq 0$ implying that the second condition in (8) is satisfied. Hence Corollary 1 applies and solution of the relaxed problem produces the optimal solution for the original one (see simulation results in Figure 2). Note that for both cases the state trajectory lies entirely in $\text{int}\mathcal{X}$.

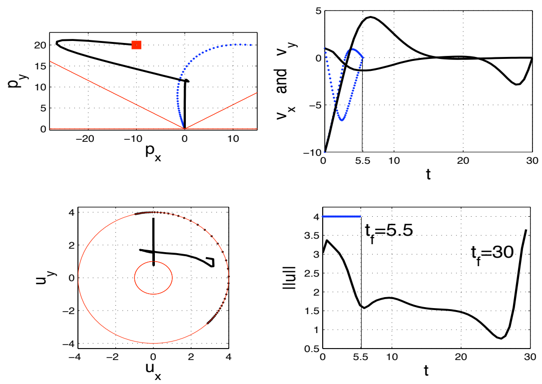

Fig. 2. *Minimum Distance Problem. Solid lines indicate solution for $t_f = 30$, and dotted lines for $t_f = 5.5$. The square in $p_x - p_y$ plot denotes z_0. The red lines in $p_x - p_y$ plot indicates the boundaries of the set of feasible states projected on $p_x - p_y$ space, and the circular annulus in $u_x - u_y$ plot indicates the set of feasible controls.*

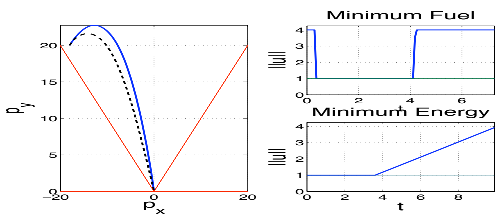

Fig. 3. *Minimum fuel versus minimum energy solutions: Dashed line is for the minimum fuel state trajectory. The control input for minimum fuel has a maximum-minimum-maximum magnitude response, while the minimum energy control input is a smooth function of time.*

In the third set of simulations, we show a comparison of different cost metrics on the control effort, namely the fuel and energy. In the fuel minimization, we have $p = 1$ in (V) and $p = 2$ in the energy minimization. Figure 3 shows the comparison between two solutions with the same initial states where the fuel and the energy are minimized. The minimum fuel control input is either at the minimum magnitude or at the maximum magnitude, which is typical for minimum fuel solutions, and the control input for the minimum energy trajectory is a smooth function. In both cases the optimal solution of the relaxed problem produced state trajectories in $\text{int}\mathcal{X}$. Consequently, by using Corollary 2, the solutions are also optimal for the original problems.

VI. Conclusions

In this paper, we present a lossless convexification of a finite horizon optimal control problem with convex state and non-convex control constraints. We introduce a convex relaxation of the problem and show that any optimal solution of the relaxed problem, with the state trajectory strictly in the interior of the set of feasible states, is also feasible for the original problem. These results enable us to utilize polynomial time convex programming algorithms to solve the relaxed problem to global optimality to obtain optimal solutions of the original problem for a number of applications of practical importance.

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Appendix

Lemma 1: Consider the system: $\dot{\lambda}(t) = -A(t)^T\lambda(t)$, $y(t) = B(t)^T\lambda(t)$, defined for $t \in [t_0, t_f]$ where $\{A(\cdot), B(\cdot)\}$ a controllable pair. If $y(t) = 0 \quad \forall t \in [t_1, t_2] \subseteq [t_0, t_f]$ then $\lambda(t) = 0 \quad \forall t \in [t_0, t_f]$.

Proof: The proof follows from standard arguments on controllability and observability of linear systems. ■

Lemma 2: Consider a hyperplane $\mathcal{E} = \{z : z = a + T\omega,\ \omega \in \mathbb{R}^m\}$ where $a \in \mathbb{R}^n$ and $\omega \in \mathbb{R}^m$ with $m < n$. We say v is orthogonal to $\mathcal{E}$ if $v^T(z_1 - z_2) = 0$ for any of z_1 and z_2 in $\mathcal{E}$. Then v is orthogonal to $\mathcal{E}$ if and only if $v \in \mathcal{E}^\perp$, which is equivalent to $T^T v = \mathbf{0}$.

Proof: The proof follows directly from the discussion in Section 2.2.1 of [4]. ■

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